PREFERENCE CONSTRUCTION FOR DATABASE QUERYING

by

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Contents

Abstract

In this dissertation, we develop methods of constructing preferences in the binary relation preference framework. This framework has been introduced recently to allow querying databases using preferences. Preferences here are strict partial order *binary relations* over objects. The framework allows for finite and infinite preference relations (represented as finite formulas).

Having a high expressive power, the binary relation framework lacks simple user- oriented methods of constructing preferences. Therefore, special query interfaces need to be developed to simplify the process of building preferences. In order to make preference construction easier, we pursue the following research directions.

First, we study the problem of attribute importance in preference relations. We propose a class of preference relations called p-skylines which extend widely used skyline preference relations with the notion of attribute importance. We show that attribute importance in p-skylines can be captured as graphs. We study properties of p-skyline relations and show methods of checking containment and dominance testing in this framework. We introduce an algorithm of computing minimal extensions of p-skyline relations.

Second, we propose to use the sets of the most preferred and the most unpreferred objects to discover importance of attributes in the underlying p-skyline relations. We study the complexity of the discovery problem and show an efficient algorithm for discovery of p-skyline relations given sets of most preferred objects.

Third, we propose to view a variant of the CP-net framework to construct preferences as preference relations. CP-nets is a well established graphical model of preference specification, widely used in AI. We introduce an original variant of CP-nets, which allows to work with infinite domains. We develop an algorithm of constructing polynomial-size formulas representing the relations induced by given CP-net instances.

Fourth, we develop a method of constructing preference relations by discarding subsets of existing preference relations. Discarding preferences is a common way of changing preferences in real life. The operation of preference contraction we develop here allows for contracting finite and infinite preference relations represented as formulas. We propose several variants of the preference contraction operator, study their properties, and introduce algorithms for their evaluation.

Chapter 1

Introduction

1.1 Motivation

User preference management is an essential part of any modern business. Knowing *what* customers like, *why* they like it, and more importantly what they *will* like in the future allows for efficient production planning, inventory stocking, and enhanced profitability. Most modern businesses use databases to store inventory related information, and the amounts of data stored grow rapidly. Thus, there is a need of efficient querying such databases with preferences.

In order to incorporate preferences into existing database query languages, the *binary relation preference framework* has been introduced independently by Kießling [Kie02] and Chomicki [Cho03]. Preferences here are *binary relations over objects*. They are required to be *strict partial orders (SPO)*: transitive and irreflexive binary relations. This framework can deal with finite as well as infinite preference relations, the latter represented using finite first order formulas which are called *preference formulas*.

EXAMPLE 1.1 *Suppose Mary wants to buy a car and her preference over cars is that she prefers newer cars and among the cars made in the same year, the cheaper one is* *preferable. This preference can be represented as the following* preference formula.

$$o_1 \succ o_2 \equiv o_1.year > o_2.year \vee o_1.year = o_2.year \wedge o_1.price < o_2.price$$

The expression above represents the fact that the car o_1 is preferred to the car o_2 iff the formula evaluates to true.

A special relational algebra operator called *winnow* [Cho03] (called *BMO* in [Kie02]) is used to compute sets of the most preferred objects in a database relation given a preference formula. An extension of SQL has been proposed to allow querying relational databases with preferences represented as binary relations [KK02].

Dealing with preferences in this framework, many existing works assume that preference relations are provided for such queries directly by users. However, such scenarios are far from being realistic, and it is hard to expect that a user can easily represent his or her preference as a preference relation. We see the following challenges here.

The first issue is that in many scenarios, formulating preferences directly by target users is impossible or hard to achieve. One reason is that a process of preference construction may take a long time which is often unacceptable. Another reason is that a user may be not clear about his or her own preferences. In such cases, *preference discovery* becomes useful. It is intended to construct an accurate model of user's preference, find hidden preferences, and avoid redundancy.

An important part of preference discovery is discovery of *attribute importance*. An attribute A is generally considered more important than another attribute B if when objects are compared by A, their values of B do not matter or matter less. In order to be able to discover the importance relationship of attributes, the preference framework has to support this notion. One class of preference relations to which this notion applies is *skyline preference relations* [BKS01]. Any such a preference relation induces equal importance of attributes. Another class of preference relations in which the notion of attribute importance is present is considered in the *preference constructor* framework [Kie04]. However, importance here is considered on the level of (possibly complex) preference relations composed in another relation. The problem of importance of *attributes* induced by such preference relations has not been studied yet. Even though a number of methods have been proposed for discovering preference relations of these classes based on user feedback [HEK03, JPL$^+$08, LwYwH$^+$08], none of them addresses discovery of attribute importance.

The second challenge is that preference relations may be rather complex. Thus, even if one has a full picture of his or her preferences in mind, it may be hard for him or her to formulate the preferences as a preference relation. Similar problem has been address in the method of constructing preferences called *CP-networks* [BBHP99]. Preferences here are defined using graphs in which every node is an attribute, and an edge from one attribute to another implies that the preference over the second attribute is conditioned on the values of the first attribute. Such conditions are expressed in the form of *conditional preference tables (CPT)* associated with every node of graph. This model exploits the *ceteris paribus* principle: given a CPT, one object is preferred to another if 1) both objects satisfy the CPT condition, 2) the first object is preferred to the second object via the CPT preference order, and 3) everything else is equal. The preference relations induced by some classes of CP-networks [BBD$^+$04] are strict partial orders. The CP-network approach is generally used outside of the scope of the binary relation model. Namely, the domains here are considered to be finite, and ad hoc algorithms are used to work with different classes of CP-networks. Some attempts have been performed to adapt CP-networks to the binary relation framework. [CS05, EK06] proposed an approach of constructing preference formulas for CP-networks. However, formulas constructed here are of exponential size in general.

The third issue is that in many cases, construction of preferences is an iterative process of their change. However, requiring a user to reconstruct the preference relation from scratch each time she changes her mind is unrealistic for practical reasons. In such scenarios, it is more natural to change preference relations step-by-step by changing only the pieces which need to be altered. Several preference change operations have been introduced in this framework: *preference revision*[Cho07b] and *equivalence adding*[BGS06]. However, these operations are limited since they allow only semantical *adding* new preferences and equivalences to existing preferences. At the same time, it is very common to discard some preferences one used to hold if the reasons for holding those preferences are no longer valid. This kind of change cannot be represented by any of the operations above.

1.2 Binary relation preference framework

In this section, we formally define a variant of the binary relation preference framework which is the scope of the current work. It is a simple and at the same time a general framework of querying databases with preferences.

Let $\mathcal{A} = \{A_1, ..., A_n\}$ be a *finite set of attributes*. Let every attribute $A_i \in \mathcal{A}$ be associated with an *infinite domain* $\mathcal{D}_{A_i}$. The domains considered here are rationals $\mathcal{Q}$ and uninterpreted constants (numerical or categorical) $\mathcal{C}$.

Let the *universe of tuples* $\mathcal{U}$ be defined as $\prod_{A_i \in \mathcal{A}} \mathcal{D}_{A_i}$. Given a subset S of $\mathcal{A}$, we denote $\prod_{A_i \in S} \mathcal{D}_{A_i}$ as $\mathcal{U}_S$. Given a tuple $o \in \mathcal{U}_S$ for some $S \subseteq \mathcal{A}$, we denote the value of the $A_i \in S$ of o as $o.A_i$. For two sets of attributes S_1 and S_2 such that $S_1 \subseteq S_2 \subseteq \mathcal{A}$, and a tuple $o \in \mathcal{U}_{S_2}$, the tuple obtained from o by leaving only the values of S_1 in it is denoted as $o.S_1$.

Given a set of attributes $S \subseteq \mathcal{A}$, we denote the set of pairs of tuples from $\mathcal{U}$ which are equal in every attribute in S as $\approx_S$, i.e.

$$\approx_S = \{(o, o') \mid o, o' \in \mathcal{U} \wedge \forall A \in S \,.\, o.A = o'.A\}.$$

Preferences in this framework are represented as *binary relations over tuples*.

DEFINITION 1.1 [Cho03] *Given a relation schema $R(A_1,\ldots,A_n)$, a relation $\succ$ is a preference relation over R if it is a subset of $\mathcal{U}\times\mathcal{U}$ and a strict partial order (SPO).*

Binary relations considered in the framework are *finite* or *infinite*. Finite binary relations are represented as sets of pairs of tuples. The infinite binary relations we consider here are *finitely representable* as *formulas*. Given a binary relation R, its formula representation is denoted F_R. The formula representation $F_{\succ}$ of a preference relation $\succ$ is called a *preference formula*.

We consider two kinds of atomic formulas here:

- *equality constraints*: $o.A_i = o'.A_i$, $o.A_i \neq o'.A_i$, $o.A_i = c$, or $o.A_i \neq c$, where o, o' are tuple variables, A_i is a $\mathcal{C}$-attribute, and c is an uninterpreted constant;

- *rational-order constraints*: $o.A_i\theta o'.A_i$ or $o.A_i\theta c$, where $\theta \in \{=,\neq,<,>,\leq,\geq\}$, o, o' are tuple variables, A_i is a $\mathcal{Q}$-attribute, and c is a rational number.

A preference formula whose all atomic formulas are equality (resp. rational-order) constraints will be called *equality* (resp. *rational order*) preference formula. If both equality and rational order constraints are used in a formula, the formula will be called *equality/rational order* formula or simply *ERO*-formula. Without loss of generality, we assume that all preference formulas are quantifier-free because ERO-formulas admit quantifier elimination.

An element of a preference relation is called *a preference*. We use the symbol $\succ$ with subscripts to refer to preference relations. We write $x \succeq y$ as a shorthand for $(x \succ y \vee x = y)$. We also say that *x is preferred to y* and *y is dominated by x according to* $\succ$ if $x \succ y$.

The two most common operations which involve preferences are

- *dominance testing*, i.e. checking if one tuples dominates another according to a given preference relation, and

- *computing the best objects*, i.e. computing sets the most preferred tuples in a given relation instance, according to a given preference relation.

To solve the former problem, one needs check if the corresponding preference formula evaluates to true for the given pair of tuples. To address the latter problem, the algebraic operator of *winnow* is used in this framework. The winnow operator picks from a given relation the set of the *most preferred tuples*, according to a given preference formula.

DEFINITION 1.2 [Cho03] *If R is a relation schema and $F_{\succ}$ a preference formula defining a preference relation $\succ$ of R, then the* winnow operator *is written as $w_{F_{\succ}}(R)$, and for every instance r of R:*

$$w_{F_{\succ}}(R) = \{o \in r \mid \neg\exists o' \in r \,.\, F_{\succ}(o', o)\}$$

Instead of the preference formula $F_{\succ}$ representing $\succ$, we can also use $\succ$ as the parameter of the winnow operator. Hence, we may write $w_{\succ}$ instead of $w_{F_{\succ}}$. In such cases, we assume that there exists a preference formula $F_{\succ}$ representing $\succ$.

1.3 Our contributions

In this section, we describe our research in the directions relevant to preference construction in the binary relation preference framework.

Extending skyline relations with attribute importance

In the area of databases, the *skyline* framework is one of the most widely used approaches of querying with preferences. Skyline preference relations are defined by the *Pareto improvement* principle: a tuple o is preferred to another one o' if o is not worse than o' in every attribute and strictly better than o' in at least one attribute. The *skyline* of a dataset is the set of the most preferred tuples according to the skyline preference relation. A well known property of skyline relations is that they induce equal importance of attributes.

In Chapter 3, we develop the *p-skyline* framework which generalizes the skyline framework by enriching it with the notion of variable attribute importance. Similarly to a skyline preference relation, a *p-skyline* preference relation is composed of preferences over attributes (also called *atomic preferences*). Every p-skyline relation is characterized by a *p-graph* which captures the difference in the importance of attributes induced by the relation. Nodes of such graphs are attributes, and edges go from more to less important attributes. A skyline preference relation is a p-skyline relation whose p-graph has no edges.

We identify the class of *full p-skyline relations*, i.e., p-skyline relations which are composed of all atomic preferences relations in a given set. We study properties of such relations. First, we show a necessary and sufficient condition for a graph to be a p-graph of a p-skyline relation. Second, we study the problems of containment and equivalence of full p-skyline relations and show that they may be reduced to simple problems of checking containment and equivalence of their p-graphs. Third, we investigate the problem of testing dominance in this framework and propose several efficient methods to solve that problem.

The next problem we explore is computing minimal extensions of full p-skyline relations. We show that given a p-skyline relation, there exists at most polynomial number of full p-skyline relations minimally extending it, all computable in polynomial time. To compute all such minimal extensions, we propose a set of simple rewriting rules which are applied to the syntax tree of a p-skyline relation.

Last, we show that preference queries in this framework may be evaluated efficiently. In particular, we show how the existing methods of skyline query evaluation may be adapted in this framework.

Feedback-based discovery of attribute importance

In Chapter 4, we study the problem of discovering user preferences in the form of p-skyline relations based on user feedback. As feedback, we propose to use sets of superior examples G (i.e, the tuples which a user likes) and inferior examples W (i.e., the tuples which a user dislikes) in a given data set O. A p-skyline relation according to which tuples in G are among the best and tuples in W are not among the best in O is called *favoring G/disfavoring W in O*. A maximal p-skyline relation favoring G/disfavoring W is called *optimal*.

First, we show that the problem of existence of a favoring/disfavoring p-skyline relation is `NP`-complete in general. Second, we prove that the problem of computing any favoring/disfavoring p-skyline relation (even an optimal one) is in general `FNP`-complete.

Next, we study restricted versions of these problems in which sets of inferior examples W are empty. We show that the problem of existence of a favoring p-skyline relation can be solved in polynomial time and may be reduced to evaluation of the skyline operator. Second, we show that the problem of computing an optimal p-skyline relation can be solved by computing a p-graph satisfying a system of constraints called *negative*. We develop an efficient polynomial time algorithm for constructing such p-skyline relations. To reduce the number of constrains used by the algorithm and hence improve its running time, we propose a set of optimization techniques. The results of experimental evaluation of the proposed algorithms are also provided in this chapter.

Graphical model to represent preference relations

In Chapter 5, we introduce a variant of the CP-net framework called *HCP-nets*. One of the issues of CP-nets we address here is that the conditionality of preferences over attributes in CP-nets does not always imply difference in attribute importance. At the same time, such a relationship between atomic preferences and the corresponding attributes may by implied by a user. It has been shown [BBD$^+$04] that this property of CP-nets is due the strictness of the *ceteris paribus* principle of comparing tuples exploited in CP-nets. In the proposed HCP-net framework, we relax this principle by requiring that children of an attribute in a conditional preference graph should not be considered when tuples are compared by that attribute. As a result, all children of an attribute in a conditional preference graph are less important.

The next deficiency of CP-nets we address in HCP-nets is the inability of CP-nets to deal with attributes with infinite domains. Since infiniteness of domains is an important property of the binary relation preference framework, we allow attribute domains to be finite or infinite in HCP-nets.

We show that the HCP-net framework has a number of interesting properties. First, we show that dominance testing in HCP-nets is in PTIME and provide several methods to solve that problem. We note that the corresponding problem for CP-nets is in PTIME for limited classes and NP-hard for a class of nets structurally close to HCP-nets. Second, we show a technique of representing orders induced by HCP-nets as polynomial size preference formulas. Last, we evaluate the proposed methods and study their scalability and running time with respect to the corresponding algorithms for CP-nets.

Constructing preference relations by discarding their subsets

In Chapter 6, we investigate the problem of constructing preference relations by changing them. In particular, we consider the operation of preference contraction – discarding a subset (called a base contractor here) of a preference relation. The preference contraction operation we propose here has two important properties: preservation of strict partial order properties in the modified preference relation and minimality of preference relation change.

We study preference contraction in the view of the scenario in which a user iteratively explores alternative ways of contracting a preference relation to find the most suitable one. We assume that a user intends to contract preferences in a *minimal* way in every iteration, i.e., discard a minimal set of preferences (including the base contractor) needed to preserve the strict partial order of the preference relation. The corresponding operator is called *minimal contraction*. This operation can be constrained by the requirement of protecting some preferences from removal (*preference protecting minimal contraction*). An important property of minimal contraction is that it may be performed in many different ways. In order to explore the effect of performing minimal contraction (preference protecting minimal contraction) in *all* possible ways and help make a decision on the way closer to the user intentions, we propose the operation of *meet contraction* (meet preference protecting contraction, respectively).

We show necessary and sufficient conditions for a subset of a preference relation to contract it minimally. We also identify a special class of base contractors – *stratifiable* base contractors. For this class, we propose a method of evaluating minimal contraction. For a subclass of stratifiable base contractors called *finitely stratifiable,* we show two algorithms for computing minimal contraction: for finite and finitely representable infinite preference relations. A method of computing meet contraction is also provided here. Additionally, we show methods of computing these operators in the presence of preference-protection constraints.

We also consider the proposed contraction operators in the context of *belief revision*. We show a variant of the preference state framework [Han95] in which operations of preference change may be computed using preference revision [Cho07b] and the contraction operators proposed here. We also study properties of this framework.

Finally, we perform an experimental evaluation of the proposed contraction operators on finite preference relations and present the results.

Chapter 2

Preliminaries

In this chapter, we review the standard notions of the partial order set theory [Sch03] and relational databases [AHV95].

2.1 Relations and graphs

We use the standard definition of binary relations. Namely, a *binary relation* R over a (finite or infinite) set S is a subset of $S \times S$. Binary relations may be *finite* or *infinite*. We write $R(x,y)$ or $xy \in R$ to denote that $(x,y) \in R$.

Here we list some typical properties of binary relations. A binary relation R is

- *irreflexive* iff $\forall x \; . \; \neg R(x,x)$,
- *asymmetric* iff $\forall x,y \; . \; R(x,y) \rightarrow \neg R(y,x)$
- *transitive* iff $\forall x,y,z \; . \; R(x,y) \wedge R(y,z) \rightarrow R(x,z)$
- *negatively transitive* iff $\forall x,y,z \; . \; \neg R(x,y) \wedge \neg R(y,z) \rightarrow \neg R(x,z)$
- *connected* iff $\forall x,y,z \; . \; R(x,y) \vee R(y,x) \vee x = y$

- a *strict partial order (SPO)* if it is irreflexive and transitive;
- a *weak order* if it is a negatively transitive strict partial order;
- a *total order* if it is a connected strict partial order.

A *weak order* R has the following property

$$\forall x, y, z \, . \, R(x,y) \rightarrow R(x,z) \vee R(z,y)$$

Let the *range* of a binary relation R be defined as

$$range_R = \{x \mid \exists y \, . \, R(x,y) \vee R(y,x)\}$$

Let the *transitive closure* of a binary relation R be denoted as $TC(R)$ and defined as

$$(x,y) \in TC(R) \text{ iff } R^m(x,y) \text{ for some } m \geq 0,$$

where

$$\begin{aligned} R^1(x,y) &\equiv R(x,y) \\ R^{m+1}(x,y) &\equiv \exists z \, . \, R(x,z) \wedge R^m(z,y) \end{aligned}$$

A finite or infinite binary relation $R \subseteq S \times S$ may be viewed as a directed graph, finite or infinite, respectively. The set S is called *the nodes of R* and denoted as $N(R)$. We say that the tuple xy is an *R-edge from x to y* if $(x,y) \in R$. A *path in R* (or an *R-path*) from x to y for an R-edge xy is a sequence of R-edges such that the start node of the first edge is x, the end node of the last edge is y, and the end node of every edge (except the last one) is the start node of the next edge in the sequence. The *length of an R-path* is the number of R-edges in the path. An *R-sequence* is the sequence of nodes participating in an R-path. The *length of an R-sequence* is the number of nodes in it.

Given a directed graph R and its node x,

- $Ch_R(x) = \{y \mid (x,y) \in R\}$ is the set of *children of x in R*,
- $Pa_R(x) = \{y \mid (y,x) \in R\}$ is the set of *parents of x in R*,
- $Desc_R(x) = \{y \mid (x,y) \in TC(R)\}$ is the set of *descendents of x in R*,
- $Anc_R(x) = \{y \mid (y,x) \in TC(R)\}$ is the set of *ancestors of x in R*,
- $Sibl_R(x) = N(R) - (Desc_R(x) \cup Anc_R(x) \cup \{x\})$ is the set of *siblings of x in R*

We also write $Desc\text{-}self_R(x)$ and $Anc\text{-}self_R(x)$ as shorthands of $(Desc_R(x) \cup \{x\})$ and $(Anc_R(x) \cup \{x\})$, respectively.

Given two nodes x and y of R and two sets of nodes X and Y of R, we write

- $R \models x \sim y$ iff $(x,y) \notin R$ and $(y,x) \notin R$;
- $R \models X \sim Y$ iff $\forall x \in X, y \in Y \,.\, R \models x \sim y$;
- $(X,Y) \in R$ iff $\forall x \in X, y \in Y \,.\, (x,y) \in R$.

2.2 Relational model

A *database schema* is a set of names of relations of fixed arity. Relation and attribute names are drawn from an infinite set of names. Every attribute of a relation is associated with a *domain*: rationals $\mathcal{Q}$ or uninterpreted constants $\mathcal{C}$. Two constants are considered to be equal if the corresponding names are the same. We use the natural interpretation of the built-in symbols $>$, $\geq$, $<$, $\leq$, and $=$ over decimals.

Throughout the document, we use the *relational algebra* language to query relational database instances. Queries in this language have the following grammar:

$$E ::= R \mid \sigma_\theta(E) \mid \phi_X(E) \mid E_1 \times E_2 \mid E_1 \cup E_2 \mid E_1 - E_2 \mid E_1 \underset{E_1.A_{i_1}=E_2.A_{j_1},\ldots,E_1.A_{i_m}=E_2.A_{j_m}}{\bowtie} E_2$$

where R is any relation name, $A_{i_1},\ldots,A_{i_m},A_{j_1},\ldots,A_{j_m}$ are attribute names, σ, ϕ, $\times$, $\cup$, $-$, $\bowtie$ are the selection, projection, Cartesian product, union, set difference, and join operators, respectively. The selection condition θ is a quantifier-free formula over the corresponding relation, and X is a list of attributes. We evaluate relation algebra expressions in the standard way.

Chapter 3

p-skyline framework

In this chapter, we propose the *p-skyline framework*. It is a generalization of the skyline approach of querying databases with preferences. Before going to details of the p-skyline framework, we describe the two frameworks it is based on: skylines and preference constructors.

3.1 Skyline framework

DEFINITION 3.1 *Let A be an attribute from the universe of attributes $\mathcal{A}$. Then* an atomic preference relation over *A is a* total order *$>_A$ which is a subset of $\mathcal{D}_A \times \mathcal{D}_A$.*

Given a set of atomic preference relations $\mathcal{H} = \{>_{A_1}, \ldots, >_{A_n}\}$ for every attribute in $\mathcal{A}$, *the skyline preference relation* for $\mathcal{H}$ is denoted as $sky_{\mathcal{H}}$. The relation $sky_{\mathcal{H}}$ represents the *Pareto improvement principle*. Namely, a tuple o is preferred to another tuple o' according to $sky_{\mathcal{H}}$ if and only if

1. for every attribute $A_i \in \mathcal{A}$, we have $o.A_i >_{A_i} o'.A_i$ or $o.A_i = o'.A_i$. In other words, o is not worse than o' according to every attribute, and

2. for some attribute $A_i \in \mathcal{A}$, we have $o.A_i >_{A_i} o'.A_i$. In other words, o is strictly preferred to o' according to at least one attribute.

Given a set of tuples r, its *skyline* is the set of the most preferred tuples in r according to $sky_{\mathcal{H}}$. In other words, the skyline of r is the result of $w_{sky_{\mathcal{H}}}(r)$. The corresponding winnow query in the skyline framework is called the *skyline query*.

It is known that skyline preference relations are strict partial orders. They are representable using preference formulas if the corresponding atomic preference relations are. A number of optimization algorithms have been developed to compute skyline queries. We discuss some of them in the related work.

3.2 Preference constructor framework

The preference constructor framework has been proposed in [Kie02]. Preference relations here are strict partial order relations constructed from a fixed set of *base preference constructors* using a set of *operators*.

DEFINITION 3.2 *Let A be an attribute in $\mathcal{A}$. Then a* base preference constructor *is a tuple $(A, >_A)$, where $>_A$ is a strict partial order over $\mathcal{D}_A$.*

Various base preference constructors are defined here for categorical as well as numerical domains. To construct a complex preference relation from base preference constructors, the following set of operators is used: *Pareto accumulation*, which represents equal importance of the preference relations being combined; *prioritized accumulation*, which represents a difference in the importance of the preference relations being combined; *intersection* and *disjoint union*, which are used to aggregate preference relations. Preference relations composed recursively using these operators are called *accumulational*.

DEFINITION 3.3 *Let $Var(\succ)$ be the set of all relevant attributes. A relation $\succ$ is an* accumulational relation *iff*

- $\succ$ *is induced by a base preference constructor* $(A, >_A)$

  $$\succ \; = \; \{(o,o') \mid o,o' \in \mathcal{U} \;.\; o.A >_A o'.A\}.$$

  Then $Var(\succ) = \{A\}$.

- $\succ$ *is a* prioritized accumulation *of two accumulational relations* $\succ_1$ *and* $\succ_2$ *(denoted as* $\succ_1 \;\&\; \succ_2$*), defined as*

  $$\succ \equiv \succ_1 \cup (\approx_{Var(\succ_1)} \;\cap\; \succ_2) \tag{3.1}$$

  Then $Var(\succ) = Var(\succ_1) \cup Var(\succ_2)$.

- $\succ$ *is a* Pareto accumulation *of two accumulational relations* $\succ_1$ *and* $\succ_2$ *(denoted as* $\succ_1 \;\otimes\; \succ_2$*), defined as*

  $$\succ \equiv \; (\succ_1 \cap \; \approx_{Var(\succ_2)}) \;\cup\; (\succ_2 \cap \; \approx_{Var(\succ_1)}) \;\cup\; (\succ_1 \cap \succ_2) \tag{3.2}$$

  Then $Var(\succ) = Var(\succ_1) \cup Var(\succ_2)$.

- $\succ$ *is an* intersection *of two accumulational relations* $\succ_1$ *and* $\succ_2$ *defined as*

  $$\succ \equiv \succ_1 \cap \succ_2,$$

  such that $Var(\succ_1) = Var(\succ_2)$. *Then* $Var(\succ) = Var(\succ_1) = Var(\succ_2)$.

- $\succ$ *is an* disjoint union *of two accumulational relations* $\succ_1$ *and* $\succ_2$ *defined as*

  $$\succ \equiv \succ_1 \cup \succ_2,$$

  such that $Var(\succ_1) = Var(\succ_2)$ *and* $range_{\succ_1} \cap \; range_{\succ_2} = \emptyset$. *Then* $Var(\succ) = Var(\succ_1)$

$= Var(\succ_1)$.

Some properties of accumulational operators are summarized below.

PROPOSITION 3.1 [Kie02] *The operators* $\otimes$ *and* $\&$ *are associative. The operator* $\otimes$ *is commutative.*

PROPOSITION 3.2 [Kie02] *An accumulational preference relation is an SPO.*

Accumulational preference relations is a foundation of the *Preference SQL* [KK02] language of querying databases with preferences. This language extends SQL by adding the PREFERRING keyword which is used to embed preferences into SQL queries. Preferences here are accumulational relations. Every base constructor has a special keyword in this language (e.g., HIGHEST, LOWEST, AROUND). Base constructors are composed into more complex expressions using the AND keyword, representing Pareto accumulation, and the CASCADE keyword, representing prioritized accumulation. Below we show an example of such a query.

```
SELECT * FROM P
PREFERRING HIGHEST(memory) AND HIGHEST(cpu) CASCADE
LOWEST(price)
```

This query is equivalent to $w_{\succ}(P)$ where

$$\succ \equiv (\succ_{memory} \otimes \succ_{cpu}) \,\&\, \succ_{price},$$

where $\succ_{memory}$ and $\succ_{cpu}$ represent the preferences that higher values of *memory* and *cpu* are preferred, and $\succ_{price}$ represents the preference that lower values of *price* are preferred.

3.3 p-skyline relations

The skyline framework and the preference constructor framework are among the major approaches of querying databases with preferences. The former approach is very simple: a user needs to provide a set of relevant attributes and atomic preferences over them to construct a preference query. A great number of optimization algorithms have been developed for skyline queries. On the other hand, a well known limitation of this framework is that attributes in a skyline preference relation are of the same importance. Hence, preferences with different importance of attributes cannot be represented as skyline preference relations.

The latter framework has much richer expressive power than the skyline framework. In this framework, preference relations composed into another preference relation may be of the same or different importance. It has been shown [KK02] that skyline preference relations are representable in this framework. Moreover, preference queries in this framework are representable as Preference SQL statements and can be evaluated efficiently [HK05].

An important property of the preference constructor framework that it deals with the notion of importance on a high level. For instance, if $\succ = \succ_1 \otimes \succ_2$ ($\succ = \succ_1 \;\&\; \succ_2$, respectively), then the preference relation $\succ_1$ is known to be as important as (more important than, respectively) $\succ_2$ in $\succ$. However, the same relationship may not hold between $\succ_1$ and $\succ_2$ after composing $\succ$ with another preference relation.

EXAMPLE 3.1 *Consider two preference relations $\succ_1$ and $\succ_2$ defined as*

$$\succ_1 \equiv \succ_{A_1} \;\&\; \succ_{A_2}$$
$$\succ_2 \equiv \succ_1 \otimes \succ_{A_2}$$

According to the semantics of $\&$, the preference relation $\succ_{A_1}$ is more important than $\succ_{A_2}$ in $\succ_1$. However, despite the fact that $\succ_1$ is used in construction of $\succ_2$, the same *importance relationship between $\succ_{A_1}$ and $\succ_{A_2}$ as in $\succ_1$ does not hold in $\succ_2$. Namely, it is easy to check that $\succ_{A_1}$ and $\succ_{A_2}$ are of equal importance in $\succ_2$.*

Moreover, the notion of *attribute importance* has not been addressed in this framework: given a preference relation, determine the importance relationship between two attributes in the preference relation.

The focus of this chapter is to develop a preference framework

1. which generalizes the skyline framework by enriching it with the notion of *attribute importance*;
2. in which *attribute importance* implied by a preference relation is intuitively captured; and
3. in which *preference queries* can still be represented as Preference SQL statements and evaluated efficiently.

Here we propose a framework which has all these properties. It is called the *p-skyline framework*. Similarly to the skyline framework, preference relations here are composed of atomic preference relations.

DEFINITION 3.4 *Let A be an attribute from the set $\mathcal{A}$ with the domain $\mathcal{D}_A$. Then an* atomic preference relation over A *is a total order $>_A$ which is a subset of $\mathcal{D}_A \times \mathcal{D}_A$ representable as a preference formula.*

To construct p-skyline relations, two operators of the preference constructor framework are used. First, Pareto accumulation is used to represent *equal* importance of the composed preference relations ($\succ_1$ and $\succ_2$ are equally important in $\succ_1 \otimes \succ_2$). Second, prioritized accumulation is used to represent *different* importance of the composed preference relations ($\succ_1$ is more importance than $\succ_2$ in $\succ_1 \;\&\; \succ_2$).

DEFINITION 3.5 *A preference relation $\succ$ is a* prioritized skyline relation *or simply* a p-skyline relation *if one of the following holds:*

1. *$\succ$ is induced by an atomic preference relation $>_A \in \mathcal{H}$ (denoted as $\succ_A$)*

$$\succ \equiv \{(o,o') \mid o,o' \in \mathcal{U} \,.\, o.A >_A o'.A\}.$$

   Then $Var(\succ) = \{A\}$.

2. *$\succ$ is a* prioritized accumulation (*see* (3.1)) *of two p-skyline relations $\succ_1$ and $\succ_2$ (denoted as $\succ_1 \ \& \ \succ_2$), where $Var(\succ_1) \cap Var(\succ_2) = \emptyset$. Then $Var(\succ) = Var(\succ_1) \cup Var(\succ_2)$.*

3. *$\succ$ is a* Pareto accumulation (*see* (3.2)) *of two p-skyline relations $\succ_1$ and $\succ_2$ (denoted as $\succ_1 \otimes \succ_2$), where $Var(\succ_1) \cap Var(\succ_2) = \emptyset$. Then $Var(\succ) = Var(\succ_1) \cup Var(\succ_2)$.*

Since accumulation operators are associative, we extend them from binary to n-ary operators.

As in the skyline framework, let the set $\mathcal{H}$ of atomic preference relations contain an atomic preference relation $>_A$ for every attribute $A \in \mathcal{A}$. We denote the set of all p-skyline relations, each composed from *all members of $\mathcal{H}$*, by $\mathcal{F}_{\mathcal{H}}$. Such relations are called *full p-skyline relations*. Further we consider only full p-skyline relations. It follows from the definition above that $Var(\succ)$ of a full p-skyline relation $\succ$ is $\mathcal{A}$.

A nice property of p-skyline relations is that they are SPO. This follows from [Kie02]. According to the next proposition, a p-skyline relation can be represented by a polynomial size formula.

PROPOSITION 3.3 *A p-skyline relation $\succ$ can be represented by a preference formula $F_\succ$ of size $|F_\succ| = O(|Var(\succ)|^2 + L_{max} \cdot |Var(\succ)|)$, where L_{max} is the length of the longest formula representing the members of $\mathcal{H}$.*

PROOF

If $\succ$ is induced by an atomic preference $>_A$, then obviously $|F_\succ| = O(L_{max})$. If $\succ = \succ_1 \ \& \ \succ_2$, then $F_\succ$ can be written using (3.1) by replacing the sets with the corresponding formulas and the set operators with the corresponding boolean operators. If $\succ = \succ_1 \otimes \succ_1$, then it can represented by the following formula

$$F_\succ(o,o') = \Big(F_{\succ_1}(o,o') \lor F_{\approx_{Var(\succ_1)}}(o,o')\Big) \land \Big(F_{\succ_2}(o,o') \lor F_{\approx_{Var(\succ_2)}}(o,o')\Big) \land \neg F_{\approx_{(Var(\succ_1) \cup Var(\succ_2))}}(o,o').$$

Note that $|F_{\approx_S}| = \Theta(|S|)$ for any set of attributes S. Hence, if $\succ$ is a Pareto or prioritized accumulation of $\succ_1$ and $\succ_2$, then

$$|F_\succ| \leq |F_{\succ_1}| + |F_{\succ_2}| + 2 \cdot (|Var(\succ_1)| + |Var(\succ_1)|) + T_1$$

for a constant T_1. By induction in the size of $F_\succ$, it can be shown that

$$|F_\succ| \leq T_2 \cdot (|Var(\succ)|^2 + L_{max} \cdot |Var(\succ)|)$$

for $T_2 \geq (2 + \frac{T_1}{2})$. Moreover, it is easy to check that a p-skyline relation whose size is $\Theta(|Var(\succ)|^2 + L_{max} \cdot |Var(\succ)|)$ may be constructed by composing atomic preference relations using only Pareto accumulation. $\square$

A key property of the p-skyline framework is that a skyline relation is a full p-skyline relation. Recall that given a set of atomic preference relations $\mathcal{H}$, $sky_{\mathcal{H}}$ is a skyline relation over the set of atomic preferences $\mathcal{H}$. It can be easily checked that

$$sky_{\mathcal{H}} \equiv \succ_{A_1} \otimes \ldots \otimes \succ_{A_n},$$

where $\succ_{A_1}, \ldots, \succ_{A_n}$ are the preference relations induced by the members $>_{A_1}, \ldots, >_{A_n}$ of

$\mathcal{H}$.

We note that the p-skyline framework is defined as a *restriction* of the preference constructor framework described in Section 3.2. These frameworks have the following distinctions:

1. in the preference constructor framework, the orders induced by *base constructors* are SPO. The corresponding atomic preference relations in the p-skyline framework are total orders. This allows us to keep the p-skyline framework similar to the skyline framework, where atomic preference relations are total orders;

2. in the p-skyline framework, we use only two operators for preference relation construction: Pareto accumulation and prioritized accumulation. As we show further, they are sufficient to express variable attribute importance in p-skyline relations;

3. for every attribute, an atomic preference relation over it appears once in a p-skyline relation. As we show further, this allows for a simple graphical method of capturing difference in the importance of attributes induced by p-skyline relations.

3.3.1 Syntax tree representation

Dealing with p-skyline relations, it is natural to represent them as *p-skyline syntax trees*.

DEFINITION 3.6 *A* p-skyline syntax tree $T_{\succ}$ *of a p-skyline relation* $\succ$ *is an ordered rooted tree representing the syntactic structure of the expression (in terms of accumulation operators and p-skyline relations induced by atomic preference relations) defining* $\succ$.

Every *non-leaf node* of a p-skyline syntax tree is labeled with an accumulation operator and corresponds to the result of applying the operator to the p-skyline relations represented by its children from left to right. Every leaf node of a p-skyline syntax tree is labeled with an attribute $A \in \mathcal{A}$ and corresponds to the p-skyline relation induced by the atomic preference relation $>_A \in \mathcal{H}$. Given a node C of a syntax tree, we denote the i-th child node of C in the syntax tree as $C[i]$. The syntax tree representation of p-skyline relations is widely used in Section 3.6 for computing minimal extensions of a p-skyline relation.

A p-skyline syntax tree is called *normalized* if every non-leaf node is labeled differently from its parent. Intuitively, a normalized p-skyline syntax tree represents the expression representing a p-skyline relation in which all parentheses nonessential due to associativity of $\otimes$ and $\&$ are removed. Clearly, for every p-skyline relation, there is a normalized p-skyline syntax tree which may be constructed in polynomial time in the size of the original tree. To do that, one needs to find all occurrences of syntax tree nodes C_1 and their children C_2 such that C_1 and C_2 have the same label. After that, C_2 has to be removed from the child list of C_1, and the list of children of C_2 has to be added to the child list of C_1 in place of C_2. This procedure is summarized in Algorithm 3.1. The function `normalizeTree` takes the root node of an unnormalized tree and performs the tree normalization. Algorithm 3.1 is trivial and provided here solely for the sake of completeness. We use Algorithm 3.1 in Chapter 4 in the algorithm for computing an optimal favoring/disfavoring p-skyline relation.

Algorithm 3.1 normalizeTree(C)

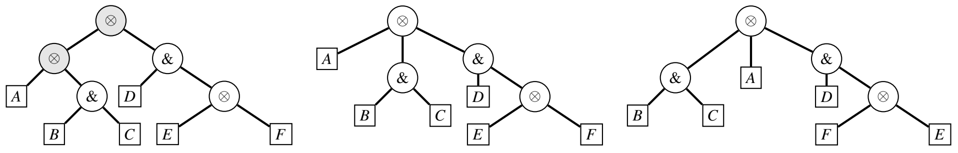
```
if $C$ is a leaf node then
  return
end if
for $i$ from the number of children of $C$ down to 1 do
  normalizeTree($C[i]$)
  if the types of $C$ and $C[i]$ are the same then
    $m :=$ the number of children in $C[i]$
    for $j$ from 1 to $m$ do
      Insert the node $C[i][j]$ as the $(i+j)$-th child to $C$
    end for
    Drop the $i$-th child of $C$
  end if
end for
```

We note that a normalized syntax tree is not unique for a p-skyline relation. That is due to commutativity of $\otimes$ (Proposition 3.1).

EXAMPLE 3.2 *Let a p-skyline relation be*

$$\succ = (\succ_A \otimes (\succ_B \; \& \succ_C)) \otimes (\succ_D \; \& (\succ_E \otimes \succ_F))$$

An unnormalized p-skyline syntax tree of $\succ$ is shown in Figure 3-1(a). Two normalized p-skyline syntax trees of $\succ$ are shown in Figures 3-1(b) and 3-1(c).



(a) Unnormalized (b) Normalized (c) Equivalent normalized

Figure 3-1: *p-skyline syntax trees of $\succ$*

Every node of a p-skyline syntax tree is itself a root of another p-skyline syntax tree. Let us associate with every node C of a p-skyline syntax tree the set $Var(C)$ of attributes which are descendants of C in the parse tree. Essentially, $Var(C)$ corresponds to $Var(\succ_C)$ introduced in Definition 3.5, where $\succ_C$ is the p-skyline relation represented by the tree with the root node C.

3.4 Attribute importance in p-skyline relations

Recall that the accumulation operators used to construct a p-skyline relation capture the importance relationships of the composed preference relations. In this section, we show how to determine the relative importance of *atomic preference relations* in a p-skyline relation. Intuitively, that corresponds to the relative importance of the corresponding *attributes*. We use the notion of $(W, \mathcal{H})$*-structures* to show that.

Essentially, the $(W, \mathcal{H})$-structure representation of a preference relation is a method of decomposing it into *dimensions* which are atomic preference relations. This decomposition shows which atomic preferences (or the corresponding attributes) are *less important* than a given atomic preference (or the corresponding attribute) in a preference relation.

The notion of $(W, \mathcal{H})$-structure is based on the set of atomic preference relations $\mathcal{H}$ and a function $W = \{W_A : A \in \mathcal{A}\}$ mapping $\mathcal{A}$ to subsets of $\mathcal{A}$.

DEFINITION 3.7 *Let W and $\mathcal{H}$ be as discussed above and such that for every $A \in \mathcal{A}$, $A \notin W_A$. Then the* $(W, \mathcal{H})$-structure *is a tuple $(W, \mathcal{H})$, and the* relation induced by $(W, \mathcal{H})$ *is*

$$\succ_{(W,\mathcal{H})} = TC\left(\bigcup_{A \in \mathcal{A}} p_A\right),$$

where

$$p_A \equiv \{(o_1, o_2) \mid o_1.A >_A o_2.A\} \cap \approx_{\mathcal{A}-\{W_A \cup \{A\}\}},$$

$>_A$ *is the atomic preference relation for A in $\mathcal{H}$.*

p_A here may be viewed as a “projection” of a preference relation $\succ_{(W,\mathcal{H})}$ to a “dimension” which is a preference relation over A. Such a projection defines the attributes $\mathcal{A} - (W_A \cup \{A\})$ whose values are important when tuples are compared by A. The values of the rest attributes W_A are not considered, i.e. they are *less important* than A. Because of that property, the function W is also referred as *the attribute importance function*.

Let a tuple o dominate a tuple o' according to the relation $\succ_{(W,\mathcal{H})}$ induced by $(W, \mathcal{H})$. By Definition 3.7, that is possible if and only if there exist a sequence of tuples $\Sigma_{o,o'} = (o_1, o_2, \ldots, o_m, o_{m+1})$ such that $o_1 = o, o_{m+1} = o'$, and a sequence of attributes $\Psi_{o,o'} = (A_{i_1}, \ldots, A_{i_m})$ such that

$$p_{A_{i_1}}(o_1, o_2), \ldots, p_{A_{i_m}}(o_m, o_{m+1})$$

Then the pair $(\Sigma_{o,o'}, \Psi_{o,o'})$ is called a *derivation sequence* for $o \succ_{(W,\mathcal{H})} o'$. Given a pair of tuples, the corresponding derivation sequence is not unique in general.

A useful property of p-skyline relations is that they can be represented as structures $(W, \mathcal{H})$. The next theorem shows a relationship between p-skyline relations and relations induced by $(W, \mathcal{H})$-structures.

THEOREM 3.1 *For every full p-skyline relation $\succ \in \mathcal{F}_{\mathcal{H}}$, there exists a $(W, \mathcal{H})$-structure which induces a relation $\succ_{(W,\mathcal{H})}$ equivalent to $\succ$. Moreover,*

1. *if $\succ$ is induced by an atomic preference $>_A$, then $W_A = \emptyset$*

2. *if $\succ\ =\ \succ_1\ \otimes\ \succ_2$, then*

$$W_A = \begin{cases} W_A^1, & \text{if } A \in Var(\succ_1) \\ W_A^2, & \text{if } A \in Var(\succ_2) \end{cases}$$

3. *if $\succ\, = \,\succ_1\ \&\ \succ_2$, then*

$$W_A = \begin{cases} W_A^1 \cup Var(\succ_2), & \text{if } A \in Var(\succ_1) \\ W_A^2, & \text{if } A \in Var(\succ_2) \end{cases}$$

for $(W^1, \mathcal{H})$ and $(W^2, \mathcal{H})$ inducing relations equivalent to $\succ_1$ and $\succ_2$.

PROOF

See Appendix A.

Theorem 3.1 shows how the attribute importance function W may be constructed for a p-skyline relation $\succ$. Namely, if attributes A, B are relevant to p-skyline relations $\succ_1$ and $\succ_2$ correspondingly, then according to $\succ\ =\ \succ_1\ \otimes\ \succ_2$, none of A, B is more important than the other in $\succ$. However, if $\succ\ =\ \succ_1\ \&\ \succ_2$, then A is more important than B in $\succ$. Moreover, if $\succ$ is used to construct a more complex p-skyline relation $\succ'$, the same importance relationship between A and B will hold in $\succ'$. An intuitive way of representing such attribute importance relationships is by means of a *p-graph*.

DEFINITION 3.8 *Let $\succ$ be a full p-skyline relation (i.e., a member of $\mathcal{F}_{\mathcal{H}}$) and $(W,\mathcal{H})$ be a structure which induces a relation equivalent to $\succ$. A graph $\Gamma_{\succ}$ whose nodes $N(\Gamma_{\succ})$ are $Var(\succ)$, and whose edges are defined as*

$$\Gamma_{\succ} = \{(X,Y) \mid X,Y \in Var(\succ) \wedge Y \in W_X\}$$

is the p-graph of $\succ$.

According to Definition 3.8, the set of nodes of a p-graph is equal to the set of the attributes relevant to the p-skyline relation. Edges in a p-graph go from more important to less important attributes. By Theorem 3.1, a p-graph of a p-skyline relation can be constructed recursively using the next corollary.

COROLLARY 3.1 *Let $\succ$ be a full p-skyline relation. Then the edge set of the p-graph $\Gamma_{\succ}$ of $\succ$ is*

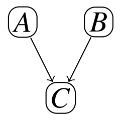
1. *empty, if $\succ$ is induced by an atomic preference;*
2. $\Gamma_{\succ_1} \cup \Gamma_{\succ_2}$, *if $\succ = \succ_1 \otimes \succ_2$;*
3. $\Gamma_{\succ_1} \cup \Gamma_{\succ_2} \cup N(\Gamma_{\succ_1}) \times N(\Gamma_{\succ_2})$, *if $\succ = \succ_1 \,\&\, \succ_2$;*

We illustrate Corollary 3.1 in the next example.

EXAMPLE 3.3 *Let $\mathcal{A} = \{A,B,C\}$, $\mathcal{H} = \{>_A, >_B, >_C\}$, and*

$$\succ_1 \equiv (\succ_A \otimes \succ_B) \,\&\, \succ_C$$
$$\succ_2 \equiv \succ_A \otimes \succ_B \otimes \succ_C$$

A relation equivalent to $\succ_1$ is induced by the structure $(W^1,\mathcal{H})$ with $W^1_A = \{C\}, W^1_B = \{C\}, W^1_C = \emptyset$. The p-graph $\Gamma_{\succ_1}$ of $\succ_1$ is shown in Figure 3-2(a).


(a) p-graph $\Gamma_{\succ_1}$

(b) p-graph $\Gamma_{\succ_2}$

Figure 3-2: *p-graphs from Example 3.3*

A relation equivalent to $\succ_2$ is induced by the structure $(W^2, \mathcal{H})$ with $W^2_A = \emptyset, W^2_B = \emptyset, W^2_C = \emptyset$. The p-graph $\Gamma_{\succ_2}$ of $\succ_2$ is shown in Figure 3-2(b).

In the previous section, we showed that the skyline relation $sky_{\mathcal{H}}$ is constructed as Pareto accumulation of all the members of $\mathcal{H}$. Hence, the next corollary holds.

COROLLARY 3.2 *The p-graph $\Gamma_{\mathrm{sky}_{\mathcal{H}}}$ of the skyline relation $\mathrm{sky}_{\mathcal{H}}$ is defined by the node set $N(\Gamma_{\mathrm{sky}_{\mathcal{H}}}) = \mathcal{A}$ and the edge set $\Gamma_{\mathrm{sky}_{\mathcal{H}}} = \emptyset$.*

The next theorem shows necessary and sufficient conditions for a directed graph to be a p-graph of some p-skyline relation.

THEOREM 3.2 (SPO+Envelope)
A directed graph Γ with the node set $\mathcal{A}$ is a p-graph of a p-skyline relation iff

1. *Γ is transitive and irreflexive, i.e. a strict partial order* (SPO), *and*
2. *Γ satisfies the* Envelope *property:*

$$\forall A, B, C, D \in \mathcal{A},\ \textit{all different}$$

$$(A,B) \in \Gamma \wedge\ (C,D) \in \Gamma \wedge (C,B) \in \Gamma \Rightarrow (C,A) \in \Gamma \vee (A,D) \in \Gamma \vee (D,B) \in \Gamma$$

As we showed above, a p-graph represents the relationships of importance between attributes induced by the corresponding p-skyline relation. Hence, the SPO properties of a p-graph are quite intuitive – they capture the rationality of the importance relationship. The

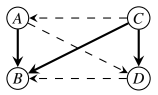


Figure 3-3: *The* `Envelope` *property*

`Envelope` property of a p-graph is due to the fact that each atomic preference relation can have only one occurrence in a p-skyline preference relation. According to that property, if a graph Γ has the three edges shown bold in Figure 3-3, then the p-graph must have at least one of the dashed edges. It is easy to check that an equivalent formulation of `Envelope` is that if Γ has the tree bold edges in Figure 3-3, then Γ must have *at least one more edge between the nodes A,B,C, and D that does not violate the SPO properties of* Γ.

To prove Theorem 3.2, we introduce the notion of the *typed partition* of a directed graph.

DEFINITION 3.9 *Let Γ be a directed graph, and Γ_1, Γ_2 be two nonempty subgraphs of Γ such that $N(\Gamma_1) \cap N(\Gamma_2) = \emptyset$ and $N(\Gamma_1) \cup N(\Gamma_2) = N(\Gamma)$. Then the pair of Γ_1, Γ_2 is a* $\sim$-partition ($\rightarrow$-partition) *of Γ if $\Gamma \models N(\Gamma_1) \sim N(\Gamma_2)$ $((N(\Gamma_1), N(\Gamma_2)) \in \Gamma$, respectively).*

The proof of Theorem 3.2 is based on lemmas 3.1 and 3.2. Lemma 3.1 establishes relationships between nodes in an `SPO+Envelope` graph, while Lemma 3.2 establishes relationships between typed partitions in such a graph.

DEFINITION 3.10 *Two nodes A and B of a directed graph Γ form a* fork *if A is different from B, and there is a node C in Γ s.t.*

$$(A,C) \in \Gamma \wedge (B,C) \in \Gamma \vee (C,A) \in \Gamma \wedge (C,B) \in \Gamma, \textit{ or}$$

$$(A,B) \in \Gamma \vee (B,A) \in \Gamma$$

Figure 3-4 shows all possible forks of two nodes A and B in a graph.

LEMMA 3.1 *Let a directed graph Γ satisfy* `SPO+Envelope`*. Then Γ has a $\sim$-partition, or any pair of nodes of Γ form a fork.*

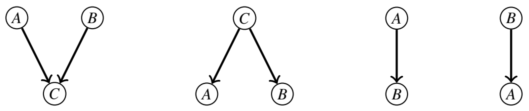


Figure 3-4: *Forks of A and B*

PROOF

For the sake of contradiction, assume Γ has no $\sim$-partition, and some pair of different nodes A and B of Γ does not form a fork, i.e.,

$$(A,B) \notin \Gamma \wedge (B,A) \notin \Gamma \wedge$$
$$\neg \exists C \in N(\Gamma) \ . \ (A,C) \in \Gamma \wedge (B,C) \in \Gamma \vee (C,A) \in \Gamma \wedge (C,B) \in \Gamma$$

Let a subgraph Γ_1 of Γ have the following set of nodes

$$N(\Gamma_1) = \{A\} \cup Pa_\Gamma(\{A\} \cup Ch_\Gamma(A)) \cup Ch_\Gamma(\{A\} \cup Pa_\Gamma(A)),$$

and the subgraph Γ_2 of Γ have the nodes $N(\Gamma_2) = N(\Gamma) - N(\Gamma_1)$. Assume that $B \in N(\Gamma_1)$. Then by definition of $N(\Gamma_1)$, we have the following four cases

1. $B \in Pa_\Gamma(A)$, contradicts the initial assumption about A and B,

2. $B \in Ch_\Gamma(A)$, same as above,

3. $B \in Pa_\Gamma(Ch_\Gamma(A))$, thus there exists a node C such that $(A,C) \in \Gamma \wedge (B,C) \in \Gamma$, and we get the same contradiction as above,

4. $B \in Ch_\Gamma(Pa_\Gamma(A))$, thus there exists a node C such that $(C,A) \in \Gamma \wedge (C,B) \in \Gamma$, and we get the same contradiction as above.

Therefore, $B \in N(\Gamma_2)$. We show that $\Gamma \models N(\Gamma_1) \sim N(\Gamma_2)$. Assume there are $C \in N(\Gamma_1)$ and $D \in N(\Gamma_2)$ such that $(C,D) \in \Gamma$. We have four cases for C:

1. $C \in Pa_\Gamma(Ch_\Gamma(A))$, i.e., there exists F such that $(A,F) \in \Gamma \wedge (C,F) \in \Gamma$. Given $(C,D) \in \Gamma$, the `Envelope` property of Γ implies that

$$(C,A) \in \Gamma \vee (A,D) \in \Gamma \vee (D,F) \in \Gamma$$

   Then the first disjunct implies that $D \in Ch_\Gamma(Pa_\Gamma(A))$, i.e., $D \in N(\Gamma_1)$; the second disjunct implies that $D \in Ch_\Gamma(A)$, i.e., $D \in N(\Gamma_1)$; the third disjunct implies that $D \in Pa_\Gamma(Ch_\Gamma(A))$, i.e., $D \in N(\Gamma_1)$. This contradicts the assumption that $D \in N(\Gamma_2)$.

2. $C \in Ch_\Gamma(Pa_\Gamma(A))$, i.e., there exists F such that $(F,A) \in \Gamma \wedge (F,C) \in \Gamma$. Then the transitivity of Γ implies $(F,D) \in \Gamma$, i.e., $D \in Ch_\Gamma(Pa_\Gamma(A))$ and thus $D \in N(\Gamma_1)$. Contradiction.

3. $C \in Ch_\Gamma(A)$, and by transitivity of Γ, we get that $D \in Ch_\Gamma(A)$ and thus $D \in N(\Gamma_1)$. Contradiction.

4. $C \in Pa_\Gamma(A)$, thus $D \in Ch_\Gamma(Pa_\Gamma(A))$ and $D \in N(\Gamma_1)$. Contradiction.

It can be shown that $(D,C) \in \Gamma$ leads to similar contradictions. Therefore, $\Gamma \models N(\Gamma_1) \sim N(\Gamma_2)$. However, this contradicts the initial assumption. □

LEMMA 3.2 *A directed graph* Γ *satisfying* SPO+Envelope *with more than one node has a* $\rightarrow$*-partition or a* $\sim$*-partition into subgraphs* $\Gamma_{\succ_1}, \Gamma_{\succ_2}$ *satisfying* SPO+Envelope.

PROOF

We assume that no $\sim$-partition of Γ exists and show that there exists a $\rightarrow$-partition. Since Γ is a finite SPO, there exists a nonempty set $Top \subseteq N(\Gamma)$ of all the nodes which have no incoming edges. If Top is a singleton, then clearly Top dominates every node in $N(\Gamma) - Top$ and we get a $\rightarrow$-partition. Assume Top is not singleton. Pick any two nodes $T_1, T_2 \in Top$. Since T_1 and T_2 have no incoming edges, Lemma 3.1 implies that there exists a node Z_1 such that $(T_1,Z_1) \in \Gamma \wedge (T_2,Z_1) \in \Gamma$. Pick some node T_k ($T_k \neq T_1, T_k \neq T_2$) from Top.

Since T_k has no incoming edges either, Lemma 3.1 implies that either T_k is a parent of Z_1 or they have a common child (which is also a child of T_1 and T_2 by transitivity of Γ). Therefore, by picking every node of Top, we can show that there exists at least one node Z which is a child of all nodes in Top. Let us denote as M the set of all the nodes such that every node in Top dominates every node in M. Above we showed that M contains at least one node.

Now let us show that if a node X is not in M then $(X,M) \in \Gamma$. Clearly, if $X \in Top$, then $(X,M) \in \Gamma$. So let $X \notin Top$. By definition of Top, there is a node $T_1 \in Top$ such that $(T_1,X) \in \Gamma$. Assume there is a node $Z \in M$ such that $(X,Z) \notin \Gamma$. By definition of M, $(T_1,Z) \in \Gamma$. Now pick some node T ($T \neq T_1$) of Top. By definition of M, $(T,Z) \in \Gamma$. Let us apply `Envelope`:

$$(T,Z) \in \Gamma \wedge (T_1,Z) \in \Gamma \wedge (T_1,X) \in \Gamma \Rightarrow (T_1,T) \in \Gamma \vee (T,X) \in \Gamma \vee (X,Z) \in \Gamma$$

The first and the last disjuncts in the right-hand-side of the expression contradict the assumptions that $T \in Top$ and $(X,Z) \notin \Gamma$. Therefore, the only choice is $(T,X) \in \Gamma$. However, T is an arbitrary node in Top. Therefore, $(Top,X) \in \Gamma$ and thus $X \in M$ by definition of M.

So if we construct a graph Γ_1 with the node set $N(\Gamma) - M$, and Γ_2 with the node set M, they will be nonempty and thus a $\rightarrow$-partition of Γ

It is easy to check that any subgraph of an `SPO+Envelope` graph satisfies that property. □

Now we return to the proof of Theorem 3.2.

PROOF OF THEOREM 3.2

By induction in the size of p-skyline relation, it is easy to show that p-graphs satisfy `SPO+Envelope`. Now we show that any directed graph satisfying `SPO+Envelope` is a p-graph of some p-skyline relation. Given such a graph, we construct the corresponding p-skyline relation recursively. If Γ contains a single node, then the corresponding p-skyline relation is induced by the atomic preference relation of the corresponding attribute-node. If Γ has more than one node, then Γ has either a $\rightarrow$-partition or a $\sim$-partition Γ_1, Γ_2 into non-empty `SPO+Envelope` subgraphs (Lemma 3.2). Pick any such a partition Γ_1, Γ_2. If it is a $\rightarrow$-partition ($\sim$-partition), then the corresponding p-skyline relation is a prioritized (Pareto, respectively) accumulation of the p-skyline relations corresponding to Γ_1 and Γ_2. This recursive construction exactly corresponds to the construction of the function W shown in Theorem 3.1. $\square$

3.5 Properties of p-skyline relations

In this section, we show some properties of p-skyline relations. These properties are used to efficiently perform some essential operations with p-skyline relations: checking equivalence and containment of two p-skyline relations and testing dominance of a tuple over another according to a p-skyline relation. Before going further, we note that since p-skyline relations are representable as formulas (Proposition 3.3), one can use the corresponding formulas to perform those operations. In this section, we show methods of performing the operations above which do not use preference formulas.

We note that so far we have introduced two graph notations for p-skyline relations: p-skyline syntax trees and p-graphs. Although these notations represent different concepts, there is a correspondence between them shown in the next proposition.

PROPOSITION 3.4 **(Syntax tree and p-graph correspondence)** *Let A and B be leaf nodes in a syntax tree $T_{\succ}$ of $\succ \in \mathcal{F}_{\mathcal{H}}$. Then $(A,B) \in \Gamma_{\Gamma}$ if and only if the least common ancestor C of A and B in $T_{\succ}$ is of type &, and A precedes B in left-to-right tree traversal.*
PROOF

$\boxed{\Leftarrow}$ Let $\succ_C$ be a p-skyline relation represented by the syntax tree with the root node C. Corollary 3.1 implies $(A,B) \in \Gamma_{\succ_C}$. Corollary 3.1 also implies that $\Gamma_{\succ_C} \subseteq \Gamma_{\succ}$.

$\boxed{\Rightarrow}$ Let $(A,B) \in \Gamma_{\succ}$. If C is of type $\&$ but B precedes A in left-to-right tree traversal, then Corollary 3.1 implies $(B,A) \in \Gamma_{\succ_C}$ and hence $(B,A) \in \Gamma_{\succ}$, which is a contradiction to SPO of $\Gamma_{\succ}$. If C is of type $\otimes$, then by Corollary 3.1, $\Gamma_{\succ_C} \models A \sim B$ and hence $\Gamma_{\succ} \models A \sim B$, which contradicts the initial assumption. □

An important observation is that the proposition above as well as Corollary 3.1 describe relationships between an *accumulation-operator expression* for a p-skyline relation and the corresponding p-graph. However, the same p-skyline relation may be represented by two or more different expressions of this type. An important question here is whether the p-graphs which correspond to these expressions are equivalent. In the next theorem we show that they really are.

THEOREM 3.3 **(p-graph uniqueness)** *Two full p-skyline relations (i.e., members of $\mathcal{F}_{\mathcal{H}}$) are equal if and only if their p-graphs are equal.*

To prove the theorem, we use the next lemma.

LEMMA 3.3 *Let for two full p-skyline relations $\succ_1, \succ_2 \in \mathcal{F}_{\mathcal{H}}$, $(W^1, \mathcal{H})$ and $(W^2, \mathcal{H})$ be two structures inducing relations equal to $\succ_1$ and $\succ_2$, respectively. Let for some $A \in \mathcal{A}$, $W^1_A - W^2_A \neq \emptyset$. Then there is a pair $o, o' \in \mathcal{U}$ such that*

$$o \succ_1 o', o \not\succ_2 o'.$$

PROOF See Appendix A.

PROOF OF THEOREM 3.3.

$\boxed{\Rightarrow}$ Any two full p-skyline relations which have the same p-graph are represented by the same structure $(W, \mathcal{H})$, by definition of p-graph. Therefore, the p-skyline relations are equivalent.

$\boxed{\Leftarrow}$ Pick any equivalent full p-skyline relations $\succ_1$ and $\succ_2$. Let the structures $(W^1, \mathcal{H})$, $(W^2, \mathcal{H})$ and the p-graphs $\Gamma_{\succ_1}$, $\Gamma_{\succ_2}$ represent $\succ_1$ and $\succ_2$, respectively. Clearly, the node sets of $\Gamma_{\succ_1}$ and $\Gamma_{\succ_2}$ are equal to $\mathcal{A}$. If their edge sets are different, then the functions W^1 and W^2 are different. Pick any $A \in \mathcal{A}$ be such that $W^1_A \neq W^2_A$. Without loss of generality, we can assume $W^1_A - W^2_A \neq \emptyset$. Lemma 3.3 implies that $\succ_1$ and $\succ_2$ are not equivalent which is a contradiction. $\square$

According to Theorem 3.3, to check equivalence of p-skyline relations, one only needs to compare their p-graphs. As the next theorem shows, containment of p-skyline relations may be also checked using their p-graph representations.

THEOREM 3.4 **(p-skyline relation containment)** *For two full p-skyline relations* $\succ_1, \succ_2 \in \mathcal{F}_{\mathcal{H}}$,

$$\succ_1 \subset \succ_2 \;\Leftrightarrow\; \Gamma_{\succ_1} \subset \Gamma_{\succ_2}.$$

PROOF

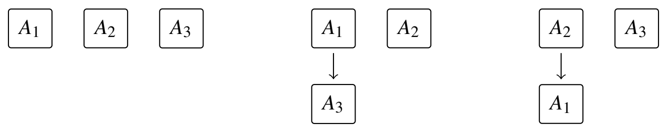
$\boxed{\Leftarrow}$ Let $(W^1, \mathcal{H})$ and $(W^2, \mathcal{H})$ be the structures which induce relations the $\succ_{(W^1,\mathcal{H})}$ and $\succ_{(W^2,\mathcal{H})}$ equivalent to $\succ_1$ and $\succ_2$ correspondingly. $\Gamma_{\succ_1} \subset \Gamma_{\succ_2}$ implies that for all $A \in \mathcal{A}$, $W^1_A \subseteq W^2_A$. Thus, $\succ_{(W^1,\mathcal{H})} \subseteq \succ_{(W^2,\mathcal{H})}$ and $\succ_1 \subseteq \succ_2$. Theorem 3.3 implies $\succ_1 \subset \succ_2$.

$\boxed{\Rightarrow}$ Let $\Gamma_{\succ_1} \not\subset \Gamma_{\succ_2}$. If $\Gamma_{\succ_1} = \Gamma_{\succ_2}$, then by Theorem 3.3, $\succ_1 = \succ_2$ which is a contradiction. Therefore, $\Gamma_{\succ_1} \neq \Gamma_{\succ_2}$, and for some A we have $W^2_A - W^1_A \neq \emptyset$. Lemma 3.3 implies $\succ_1 \not\subset \succ_2$ which is a contradiction. $\square$

Theorem 3.4 implies an important result. Recall that in Corollary 3.2 we showed that the edge set of the p-graph $\Gamma_{sky_{\mathcal{H}}}$ of the skyline preference relation $sky_{\mathcal{H}}$ is empty. Hence, the following facts are implied by Theorem 3.4.

COROLLARY 3.3 *The skyline relation* $\mathrm{sky}_{\mathcal{H}}$ *is* the least full p-skyline relation *in* $\mathcal{F}_{\mathcal{H}}$.

COROLLARY 3.4 *For any relation instance r and any full p-skyline relations* $\succ_1$, $\succ_2 \in \mathcal{F}_{\mathcal{H}}$, *such that* $\succ_2 \subset \succ_1$, *we have* $w_{\succ_1}(r) \subseteq w_{\succ_2}(r) \subseteq w_{\mathrm{sky}_{\mathcal{H}}}(r)$



(a) $\Gamma_{sky_{\mathcal{H}}}$ (b) $\Gamma_{\succ_1}$ (c) $\Gamma_{\succ_2}$

	A_1	A_2	A_3
t_1	2	1	0
t_2	1	2	0
t_3	1	0	2
t_4	1	0	0

(d) r

Figure 3-5: *Containment of p-skyline relations*

The importance of Corollary 3.4 is that if we take any full p-skyline relation $\succ \in \mathcal{F}_{\mathcal{H}}$, the corresponding preference query result will always be contained in the corresponding skyline. In real life, that means that if user preferences are modeled as p-skyline relations instead of skyline relation, the sizes of the corresponding preference query results may be smaller.

EXAMPLE 3.4 *Let $\mathcal{A} = \{A_1, A_2, A_3\}$, and for every attribute, we prefer larger values. Consider the relations* $\text{sky}_{\mathcal{H}}, \succ_1, \succ_2$

$$\begin{aligned} \text{sky}_{\mathcal{H}} &= \succ_{A_1} \otimes \succ_{A_2} \otimes \succ_{A_3} \\ \succ_1 &= (\succ_{A_1} \ \& \ \succ_{A_3}) \otimes \succ_{A_2} \\ \succ_2 &= (\succ_{A_2} \ \& \ \succ_{A_1}) \otimes \succ_{A_3} \end{aligned}$$

whose p-graphs are shown in Figures 3-5(a), 3-5(b), and 3-5(c), correspondingly. Theorems 3.4 and 3.3 imply that $\text{sky}_{\mathcal{H}} \subset \succ_1$*,* $\text{sky}_{\mathcal{H}} \subset \succ_2$*, $\succ_1 \not\subseteq \succ_2$, and $\succ_2 \not\subseteq \succ_1$. Take the relation r shown in Figure 3-5(d). Then $w_{\text{sky}_{\mathcal{H}}}(r) = \{t_1, t_2, t_3\}$, $w_{\succ_1}(r) = \{t_1, t_2\}$, and $w_{\succ_2}(r) = \{t_2, t_3\}$.*

In the next theorem, we show how one can test dominance of a tuple over another according to a p-skyline relation without using the a preference formula.

THEOREM 3.5 **(p-skyline dominance testing)** *Let $\succ$ be a full p-skyline relation, and $o, o' \in \mathcal{U}$ such that $o \neq o'$. Let also $BetterIn(o_1, o_2) = \{A \mid o_1.A >_A o_2.A\}$, $Diff(o_1, o_2)$ be the set of attributes in which o_1 and o_2 are different, and $Top(o_1, o_2)$ be the set of the topmost elements of $Diff(o_1, o_2)$ in $\Gamma_{\succ}$. Then the following statements are equivalent:*

1. *$o \succ o'$;*
2. *$BetterIn(o, o') \supseteq Top(o, o')$;*
3. *$Ch_{\Gamma_{\succ}}(BetterIn(o, o')) \supseteq BetterIn(o', o)$.*

The intuition beyond the theorem above is a follows. Method 2 of testing dominance implies that o is preferred to o' if and only if o is preferred to o' according to all the most important attributes in which these tuples are different. Method 3 of testing dominance says that o is preferred to o' if and only if for every attribute in which o' is better than o, there is a more important attribute in which o is better than o'.

PROOF

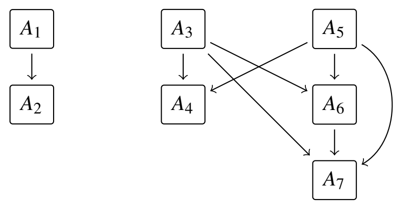
Let the structure $(W, \mathcal{H})$ induce a relation equivalent to $\succ$, i.e.

$$\succ = \succ_{(W,\mathcal{H})} = TC\left(\bigcup_{A \in \mathcal{A}} p_A\right)$$

where

$$p_A \equiv \{(o_1, o_2) \mid o_1 \succ_A o_2\} \cap \approx_{\mathcal{A} - W_A - \{A\}} .$$

$\boxed{1 \Leftrightarrow 3}$ Let $Ch_{\Gamma_{\succ}}(BetterIn(o, o')) \supseteq BetterIn(o', o)$. It is easy to check that the sequence $(\Sigma_{o,o'}, \Psi_{o,o'})$ constructed as follows is a derivation sequence for $o \succ_{(W,\mathcal{H})} o'$. Let $\Psi_{o,o'}$ be a sequence of all attributes in $BetterIn(o, o')$. For every $A_i \in \Psi_{o,o'}$, let $o_i.A_i, o_{i+1}.A_i$ be equal to the values of A_i in o and o' correspondingly. Let the values of the attributes



(a) $\Gamma_{\succ}$

id	A_1	A_2	A_3	A_4	A_5	A_6	A_7
t_1	1	1	1	1	1	1	1
t_2	2	0	1	0	2	1	0
t_2	2	0	1	0	1	2	0

(b) Tuples to compare

Figure 3-6: *Theorem 3.5 for dominance testing*

$BetterIn(o',o) \cap W_{A_i}$ be equal to those of o in o_i and those of o' in o_{i+1}. Let the values of the other attributes in o_i, o_{i+1} be equal.

Now assume $Ch_{\Gamma_{\succ}}(BetterIn(o,o')) \not\supseteq BetterIn(o',o)$. Thus, the set $BetterIn(o',o) - Ch_{\Gamma_{\succ}}(BetterIn(o,o'))$ is nonempty. Similarly to the proof of Lemma 3.3, it can be shown that no derivation sequence exists for $o \succ_{(W,\mathcal{H})} o'$.

$\boxed{2 \Leftrightarrow 3}$ 2 implies 3 by definition of $Top(o,o')$. Prove that 3 implies 2. Assume that 3 holds but $\exists A \in Top(o,o') - BetterIn(o,o')$. Since $>_A$ is a total order, $A \in BetterIn(o',o)$. Then 3 implies that $A \notin Top(o,o')$, which is a contradiction. $\square$

EXAMPLE 3.5 *Let $\mathcal{A} = \{A_1, \ldots, A_7\}$, and for every attribute, we prefer larger values. Let a p-skyline relation $\succ$ be represented by the p-graph shown in Figure 3-6(a). Take the tuples t_1, t_2, and t_3 shown in Figure 3-6(b). $BetterIn(t_1,t_2) = \{A_2,A_4,A_7\}$, $BetterIn(t_2,t_1) = \{A_1,A_5\}$, $Diff(t_1,t_2) = \{A_1,A_2,A_4,A_5,A_7\}$, and $Top(t_1,t_2) = \{A_1,A_5\}$. Hence, $t_2 \succ t_1$ and $t_1 \not\succ t_2$. $BetterIn(t_1,t_3) = \{A_2,A_4,A_7\}$, $BetterIn(t_3,t_1) = \{A_1,A_6\}$, $Diff(t_1,t_3) = \{A_1,A_2,A_4,A_6,A_7\}$, and $Top(t_1,t_3) = \{A_1,A_4,A_6\}$. Thus, $t_3 \not\succ t_1$ and $t_1 \not\succ t_3$.*

In Theorem 3.2, we showed that p-graphs satisfy SPO+Envelope, where the property Envelope was formulated in terms of distinct nodes. However, it is often necessary to deal with *sets* of p-graph nodes. The next theorem generalizes the Envelope property to sets of nodes of a p-graph. We call this property GeneralEnvelope.

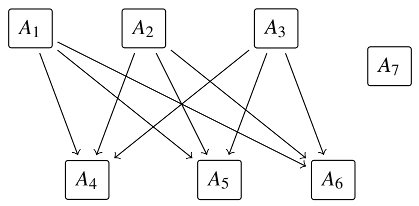


Figure 3-7: *The* `GeneralEnvelope` *property*

`THEOREM 3.6` (`GeneralEnvelope`) *Let $\succ$ be a p-skyline relation with the p-graph $\Gamma_{\succ}$. Let also $\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D}$ be disjoint node sets of $\Gamma_{\succ}$. Let the subgraphs of $\Gamma_{\succ}$ induced by those node sets be singletons or unions of at least two disjoint subgraphs. Then*

$$(\mathbf{A},\mathbf{B}) \in \Gamma_{\succ} \wedge (\mathbf{C},\mathbf{D}) \in \Gamma_{\succ} \wedge (\mathbf{C},\mathbf{B}) \in \Gamma_{\succ} \Rightarrow$$
$$(\mathbf{C},\mathbf{A}) \in \Gamma_{\succ} \vee (\mathbf{A},\mathbf{D}) \in \Gamma_{\succ} \vee (\mathbf{D},\mathbf{B}) \in \Gamma_{\succ}$$

`PROOF` See Appendix A.

Unlike `Envelope` which holds for any combination of four different nodes, the property of `GeneralEnvelope` holds for a certain class of disjoint node subsets. However, that class is quite general. For instance, $Var(\succ)$ induces disjoint subgraphs if $\succ$ is defined as Pareto accumulation of p-skyline relations.

`EXAMPLE 3.6` *Let $\mathcal{A} = \{A_1,\ldots,A_7\}$. Consider the p-graph $\Gamma_{\succ}$ (Figure 3-7) of*

$$\succ = ((>_{A_1} \otimes >_{A_2} \otimes >_{A_3}) \ \&\ (>_{A_4} \otimes >_{A_5} \otimes >_{A_6})) \otimes >_{A_7}$$

Take $\mathbf{A} = \{A_1\}$, $\mathbf{B} = \{A_4\}$, $\mathbf{C} = \{A_2,A_3\}$, $\mathbf{D} = \{A_5,A_6\}$. Then we have

$$(\mathbf{A},\mathbf{B}) \in \Gamma_{\succ} \wedge (\mathbf{C},\mathbf{D}) \in \Gamma_{\succ} \wedge (\mathbf{C},\mathbf{B}) \in \Gamma_{\succ} \wedge (\mathbf{A},\mathbf{D}) \in \Gamma_{\succ}$$

3.6 Minimal extensions of p-skyline relations

Consider the following scenario. Let us have a full p-skyline relation $\succ$ (i.e., a member of $\mathcal{F}_{\mathcal{H}}$) and two tuples o_1 and o_2 such that $o_1 \not\succ o_2$. Assume that we have the following problem: *test if there is an extension $\succ_{ext} \in \mathcal{F}_{\mathcal{H}}$ of $\succ$ such that $o_1 \not\succ_{ext} o_2$*. In other words, we need to check if $\succ$ is a maximal full p-skyline relation according to which o_1 is not preferred to o_2. Problems of this type are considered in the next chapter where we show an approach of discovering full p-skyline relation using user feedback. One approach to solve this problem is to enumerate all extensions of $\succ$ in $\mathcal{F}_{\mathcal{H}}$ and test if o_1 dominates o_2 according to each of them. However, the number of such extensions may be quite large. Instead, one could consider only *minimal extensions* of $\succ$ in $\mathcal{F}_{\mathcal{H}}$: if according to some minimal extension $\succ_{ext}$ we have $o_1 \not\succ_{ext} o_2$, then $\succ$ is not a maximal relation satisfying the property, otherwise it is. In this section, we study the problem of computing efficiently all full p-skyline relations which are minimal extensions of a given full p-skyline relation. The formal definition of a minimal extension of a full p-skyline relation is given below.

DEFINITION 3.11 *Given a full p-skyline relation $\succ \in \mathcal{F}_{\mathcal{H}}$, a full p-skyline relation $\succ_{ext} \in \mathcal{F}_{\mathcal{H}}$ is a* full p-skyline extension of $\succ$ *if $\succ \subset \succ_{ext}$. The extension $\succ_{ext}$ is* minimal *if there is no full p-skyline relation $\succ' \in \mathcal{F}_{\mathcal{H}}$ such that $\succ \subset \succ' \subset \succ_{ext}$.*

We showed in Theorem 3.4 that given any full p-skyline relation $\succ$, a full p-skyline extension $\succ_{ext}$ of $\succ$ may be obtained by computing an extension $\Gamma_{\succ_{ext}}$ of the p-graph $\Gamma_{\succ}$. Hence, the problem of computing a minimal full p-skyline extension of a p-skyline relation can be reduced to the problem of finding a minimal set of edges that when added to $\Gamma_{\succ}$ form a graph satisfying `SPO+Envelope`. However, it is not clear how to find such a minimal set of edges efficiently. Moreover, if one wants to compute *all* minimal full p-skyline extensions of a p-skyline relation, then *all* minimal sets of such edges have to be computed. Another problem of such an approach is that it may be the case that p-skyline relations in a particular application are represented as accumulation operator expressions. In this case, one needs to convert a p-skyline syntax tree to a p-graph, compute its minimal `SPO+Envelope` extension, and then convert the p-graph back to a p-skyline syntax tree.

The method of computing all minimal extensions we propose here operates directly with p-skyline relation expressions represented as p-skyline syntax trees. In particular, we show a set of transformation rules of p-skyline syntax trees such that every unique application of a rule from this set results in a unique minimal full p-skyline extension of the original p-skyline relation. If *all* minimal full p-skyline extensions of a p-skyline relation are needed, then one needs to apply to the syntax tree *every* rule in every possible way.

The transformation rules are shown in Figure 3-9. On the left hand side, we show a part of the syntax tree of an original p-skyline relation. On the right hand side, we show how the corresponding part is modified in the resulting relation. We assume that the rest of the syntax tree is left unchanged. All the transformation rules perform some manipulations with two children C_i and C_{i+1} of a $\otimes$ -node of a syntax tree. For the sake of simplicity, these nodes are shown as consecutive children. However, in the rules we assume that they may be any pair of children nodes of the same $\otimes$ -node.

Let us denote the original relation as $\succ$ and the relation obtained as a result of applying one of the transformation rules as $\succ_{ext}$. It can be easily checked by Corollary 3.1 that all the rules only *add* edges to the p-graph of the original preference relation and hence extend the p-skyline relation.

`OBSERVATION 3.1` *If $T_{\succ'}$ is obtained from $T_{\succ}$ using any of $Rule_1, \ldots, Rule_4$, then $\Gamma_{\succ} \subset \Gamma_{\succ'}$. Moreover,*

- *if $T_{\succ'}$ is a result of $Rule_1(T_{\succ}, C_i, C_{i+1})$, then*

$$\Gamma_{\succ'} - \Gamma_{\succ} = \{XY \mid X \in Var(N_1), Y \in Var(C_{i+1})\}$$

- *if $T_{\succ'}$ is a result of $Rule_2(T_{\succ}, C_i, C_{i+1})$, then*

$$\Gamma_{\succ'} - \Gamma_{\succ} = \{XY \mid X \in Var(C_{i+1}), Y \in Var(N_m)\}$$

- *if $T_{\succ'}$ is a result of $Rule_3(T_{\succ}, C_i, C_{i+1})$, then*

$$\Gamma_{\succ'} - \Gamma_{\succ} = (C_i, C_{i+1})$$

- *if $T_{\succ'}$ is a result of $Rule_4(T_{\succ}, C_i, C_{i+1}, s, t)$ for $s \in [1, n-1], t \in [1, m-1]$, then*

$$\begin{aligned}\Gamma_{\succ'} - \Gamma_{\succ} =& \{XY \mid X \in \bigcup_{p \in 1...s} Var(N_p), Y \in \bigcup_{q \in t+1...n} Var(M_q)\} \cup \\ & \{XY \mid X \in \bigcup_{p \in 1...t} Var(M_p), Y \in \bigcup_{q \in s+1...m} Var(N_q)\}\end{aligned}$$

We note that every & - and $\otimes$ -node in a p-skyline syntax tree has to have at least two child nodes. This is because the operators & and $\otimes$ must have at least two arguments. However, as a result of a transformation rule application, some & - and $\otimes$ -nodes may have only one child node. These nodes are:

1. R' if $k = 2$ for $Rule_1, Rule_2, Rule_3, Rule_4$;
2. R'_2 if $m = 2$ for $Rule_1, Rule_2$;
3. R'_3 or R'_5 if $s = 1$ or $s = m - 1$, respectively, for $Rule_4$;
4. R'_4 or R'_6 if $t = 1$ or $t = n - 1$, respectively, for $Rule_4$.

In such cases, we remove the nodes with a single child and connect the child directly to the parent, as shown in Figure 3-8.

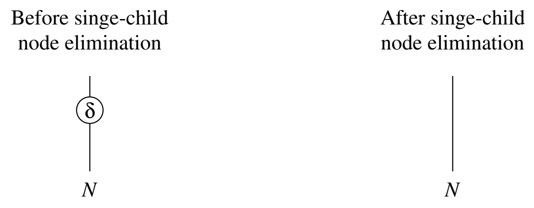

Before singe-child node elimination
δ
N
After singe-child node elimination
N

Figure 3-8: *Single-child node elimination ($\delta \in \{ \& , \otimes \}$)*

THEOREM 3.7 **(minimal full p-skyline extension)** *Let $\succ \in \mathcal{F}_{\mathcal{H}}$, and $T_{\succ}$ be a normalized syntax tree of $\succ$. Then $\succ_{ext}$ is a* minimal full p-skyline extension *of $\succ$ if and only if the syntax tree $T_{\succ_{ext}}$ of $\succ_{ext}$ is obtained from $T_{\succ}$ by a single application of a rule from $Rule_1, \ldots, Rule_4$.*

In order to prove Theorem 3.7 we introduce the notions of *frontier nodes* and *top* and *bottom* components in syntax tree.

DEFINITION 3.12 *The* top (bottom) *component of a p-skyline relation $\succ$ is*

1. *$\succ$ ($\succ$, respectively), if $\succ$ is induced by an atomic preference relation;*
2. *$\succ_1$ ($\succ_m$, respectively), if*

$$\succ = \succ_1 \ \& \ \ldots \ \& \ \succ_m$$

Note that the notions of top and bottom components are undefined for p-skyline relations defined as Pareto accumulation of p-skyline relations.

DEFINITION 3.13 *Let $T_{\succ}$ be a normalized syntax tree of a full p-skyline relation $\succ$. Let also C_1 and C_2 be two child nodes of a $\otimes$-node C in $T_{\succ}$. Let $\succ_{ext}$ be a full p-skyline extension of $\succ$, and the subgraphs of $\Gamma_{\succ}$ and $\Gamma_{\succ_{ext}}$ induced by $Var(C_1)$ and $Var(C_2)$ be equal. Let $X \in Var(C_1)$, $Y \in Var(C_2)$ be such that*

$$(X,Y) \in \Gamma_{\succ_{ext}}.$$

Then (C_1, C_2) *is the* frontier pair of $T_{\succ}$ w.r.t. $T_{\succ_{ext}}$.

Given a frontier pair (C_1, C_2) of $T_{\succ}$ w.r.t. $T_{\succ_{ext}}$, note that $\Gamma_{\succ} \models Var(X) \sim Var(Y)$ by Proposition 3.4. By definition, a p-skyline relation is constructed in a recursive way: a higher level relation is defined in terms of lower level relations. Hence, the intuition beyond the frontier pair is as follows. When $\succ$ and $\succ_{ext}$ are constructed, the lower-level relations $\succ_{C_1}$ and $\succ_{C_2}$ are present in both $\succ$ and $\succ_{ext}$. However, the next-level relations aggregating $\succ_{C_1}$ and $\succ_{C_2}$ in $\succ$ and $\succ_{ext}$ are different since $\Gamma_{\succ_{ext}}$ has an edge from a member of $Var(\succ_{C_1}$) to a member of $Var(\succ_{C_2})$ which is not present in $\Gamma_{\succ}$. The next lemma shows some properties of frontier pairs.

LEMMA 3.4 *Let $\succ_{ext}$ be a full p-skyline extension of $\succ \in \mathcal{F}_{\mathcal{H}}$, and $T_{\succ}$ be a normalized syntax tree of $\succ$. Let also (C_1, C_2) be a frontier pair of $T_{\succ}$ w.r.t. $T_{\succ_{ext}}$. Denote the the top and the bottom components of C_1 as A_1, B_1, and the top and the bottom components of C_2 as A_2, B_2. Then*

$$(Var(A_1), Var(B_2)) \in \Gamma_{\succ_{ext}} \vee (Var(A_2), Var(B_1)) \in \Gamma_{\succ_{ext}}$$

PROOF See Appendix A.

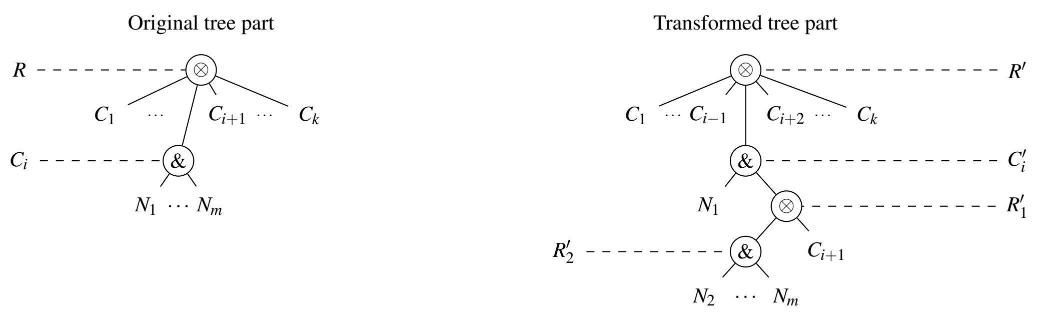
PROOF OF THEOREM 3.7

$\boxed{\Rightarrow}$ Let $\succ_{ext}$ be a minimal extension of $\succ$. We show here that there is $\succ' \in \mathcal{F}_{\mathcal{H}}$ obtained using a transformation rule $Rule_1, \ldots, Rule_4$ such that

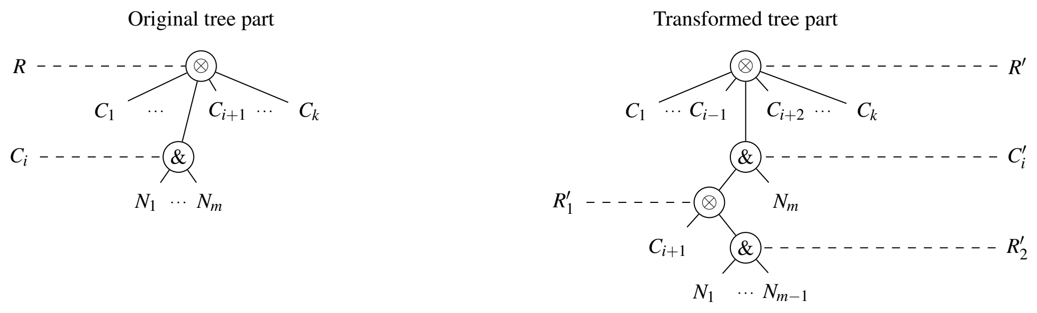
$$\succ \subset \succ' \subseteq \succ_{ext} . \tag{3.3}$$

By the minimal extension property of $\succ_{ext}$ that implies $\succ' = \succ_{ext}$.

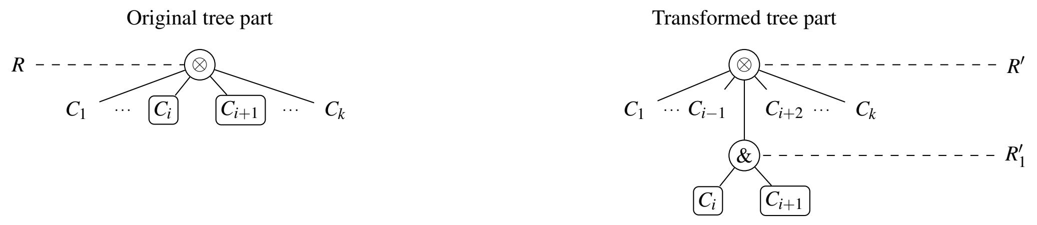
Theorem 3.4 implies that there are X, Y such that $(X, Y) \in \Gamma_{\succ_{ext}} - \Gamma_{\succ}$. Let (C_1, C_2) be a frontier pair of $T_{\succ}$ w.r.t. $T_{\succ_{ext}}$ such that $X \in Var(C_1)$ and $Y \in Var(C_2)$. Lemma 3.4 implies



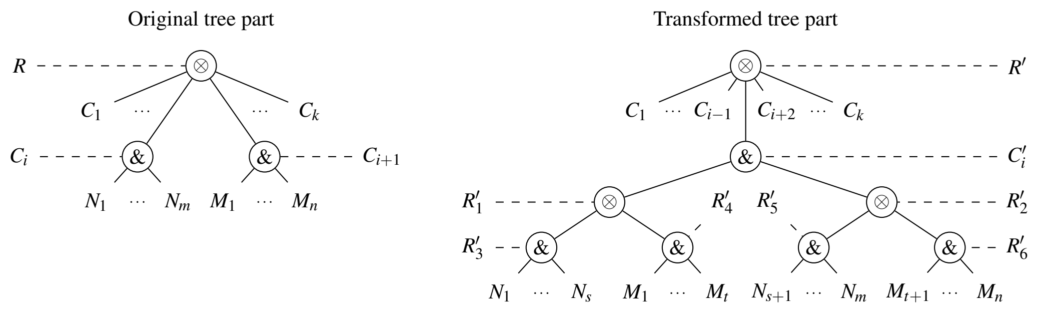
(a) $Rule_1(T_{\succ}, C_i, C_{i+1})$



(b) $Rule_2(T_{\succ}, C_i, C_{i+1})$



(c) $Rule_3(T_{\succ}, C_i, C_{i+1})$


Original tree part
Transformed tree part
R
C_1 ⋯ ⋯ C_k
C_i
C_{i+1}
N_1 ⋯ N_m M_1 ⋯ M_n
R'
C_1 ⋯ C_{i-1} C_{i+2} ⋯ C_k
C'_i
R'_1
R'_4 R'_5
R'_2
R'_3
R'_6
N_1 ⋯ N_s M_1 ⋯ M_t N_{s+1} ⋯ N_m M_{t+1} ⋯ M_n

(d) $Rule_4(T_{\succ}, C_i, C_{i+1}, s, t)$

$\boxed{C_i}$ - leaf node

C_i - leaf or non-leaf node

Figure 3-9: *Syntax tree transformation rules*

that

$$(Var(A_1), Var(B_2)) \in \Gamma_{\succ_{ext}} \vee (Var(A_2), Var(B_1)) \in \Gamma_{\succ_{ext}} \tag{3.4}$$

for the top A_1, A_2 and the bottom B_1, B_2 components of C_1 and C_2 correspondingly. Consider all possible types of C_1 and C_2. (i) Let C_1, C_2 be leaf nodes. Then $\succ'$ for which (3.3) holds may be obtained by applying $Rule_3(T_\succ, C_1, C_2)$ (if the first disjunct of (3.4) holds) or $Rule_3(T_\succ, C_2, C_1)$ (if the second disjunct of (3.4) holds). (ii) Let C_1 be a & -node and C_2 be a leaf node. Then $\succ'$ may be obtained by applying $Rule_1(T_\succ, C_1, C_2)$ (if the first disjunct of (3.4) holds) or $Rule_2(T_\succ, C_1, C_2)$ (if the second disjunct of (3.4) holds). Case (iii) when C_1 is a leaf node and C_2 is a & -node is similar to the previous case. Consider case (iv) when C_1 and C_2 are & -nodes. Let the first disjunct of (3.4) hold. The case of the second disjunct is analogous. We note that $(Var(A_1), Var(B_1)) \in \Gamma_{\succ_{ext}}$ and $(Var(A_2), Var(B_2)) \in \Gamma_{\succ_{ext}}$. This along with (3.4) is a condition for `GeneralEnvelope`:

$$\begin{aligned}(Var(A_1), Var(A_2)) \in \Gamma_{\succ_{ext}} \vee (Var(A_2), Var(B_1)) \in \Gamma_{\succ_{ext}} \vee \\ (Var(B_1), Var(B_2)) \in \Gamma_{\succ_{ext}}\end{aligned} \tag{3.5}$$

If the first disjunct of (3.5) holds, then $\succ'$ can be obtained by applying $Rule_1(T_\succ, C_1, C_2)$. If the last disjunct of (3.5) holds, then $\succ'$ can be obtained by applying $Rule_2(T_\succ, C_2, C_1)$. Let the second disjunct of (3.5) hold, i.e. $(Var(A_2), Var(B_1)) \in \Gamma_{\succ_{ext}}$. Let the child nodes of C_1 and C_2 be the sequences $(A_1 = N_1, \ldots, N_m = B_1)$ and $(A_2 = M_1, \ldots, M_n = B_2)$ correspondingly. Since C_1 and C_2 are & -nodes, $(Var(N_i), Var(N_j)) \in \Gamma_\succ$ and $(Var(M_i), Var(M_j)) \in \Gamma_\succ$ for all $i < j$. Since $\succ \subseteq \succ_{ext}$, the same edges are present in $\Gamma_{\succ_{ext}}$. Note that $(N_1, M_n) \in \Gamma_{\succ_{ext}}$. Pick every child of C_2 in its list of children from right to left and find the first index t such that $(Var(N_1), Var(M_t)) \notin \Gamma_{\succ_{ext}}$ but $(Var(N_1), Var(M_{t+1})) \in \Gamma_{\succ_{ext}}$. If no such t exists, then $(Var(N_1), Var(M_1)) \in \Gamma_{\succ_{ext}}$ and $\succ'$ may be obtained by applying $Rule_1(T_\succ, C_1, C_2)$. So assume $t \in [1, n]$. Similarly, let s be the first index such that $(Var(M_1), Var(N_s)) \notin \Gamma_{\succ_{ext}}$

but $(Var(M_1), Var(N_{s+1})) \in \Gamma_{\succ_{ext}}$. If s does not exist, then $\succ'$ may be obtained by applying $Rule_2(T_{\succ}, C_2, C_1)$. So assume $s \in [1,m]$. If both s and t are equal to 1, then $\succ'$ may be obtained using $Rule_4(T_{\succ}, C_1, C_2, s, t)$. In all other cases, it can be shown using `GeneralEnvelope` that for all $i \in [1,s], j \in [t+1,n]$ $(Var(N_i), Var(M_j)) \in \Gamma_{\succ_{ext}}$ and for all $i \in [1,t], j \in [s+1,m]$ $(Var(M_i), Var(N_j)) \in \Gamma_{\succ_{ext}}$. Hence the $Rule_4(T_{\succ}, C_1, C_2, s, t)$ may be used to construct $\succ'_{ext}$.

$\boxed{\Leftarrow}$ We showed above that any minimal full p-skyline extension $\succ_{ext}$ of $\succ$ may be computed using a single application of one of the transformation rules. Showing that any valid application of a transformation rule leads to a minimal full p-skyline extension of $\succ$ may be done by using Lemma 3.4 and a case analysis similar to the one above. □

Theorem 3.7 has two important corollaries describing properties of p-skyline relations.

COROLLARY 3.5 *Given any full p-skyline relation $\succ$ with a normalized syntax tree $T_{\succ}$, a syntax tree $T_{\succ'}$ of any its minimal full p-skyline extension $\succ'$ may be computed in time $O(|\mathcal{A}|)$.*

In Corollary 3.5, we assume the adjacency-list representation of syntax trees. The total number of nodes in a tree is linear in the number of its leaf nodes [CLRS01] which are $\mathcal{A}$. Thus the number of edges in $T_{\succ}$ is $O(|\mathcal{A}|)$. The transformation of $T_{\succ}$ using any rule requires removing $O(|\mathcal{A}|)$ and adding $O(|\mathcal{A}|)$ edges.

COROLLARY 3.6 *For any full p-skyline relation $\succ$, the number of its minimal full p-skyline extensions is $O(|\mathcal{A}|^4)$.*

The justification for Corollary 3.6 is as follows. The set of minimal-extension rules is complete due to Theorem 3.7. Every rule operates on two nodes C_i, C_{i+1} of a syntax tree. Hence, the number of such node pairs is $O(|\mathcal{A}|^2)$. $Rule_4$ also relies on some partitioning of the sequence of child nodes of C_i and C_{i+1}. The total number of such partitionings is $O(|\mathcal{A}|^2)$. As a result, the total number of different rule applications is $O(|\mathcal{A}|^4)$.

An important observation here is that the number of minimal full p-skyline extensions is *polynomial* in the number of attributes. This differs from the number of *all* extensions of a p-skyline relation. For instance, take the full p-skyline relation $sky_{\mathcal{H}}$. By Theorem 3.4, any full p-skyline relation whose p-graph is a total order of $\mathcal{A}$ is a full p-skyline extension of $sky_{\mathcal{H}}$. Theorem 3.3 implies that all p-skyline relations defined by different total-order p-graphs are different. The number of such p-graphs is $|\mathcal{A}|!$. Hence, the total number of full p-skyline extensions of $sky_{\mathcal{H}}$ is $\Omega(|\mathcal{A}|!)$.

The last property of p-skyline relations which we state here and which is related to their extensions is as follows. By Theorem 3.4, a full p-skyline extension of a p-skyline relation $\succ$ is obtained by adding edges to its p-graph. However, the total number of edges in a p-graph is at most $O(|\mathcal{A}|^2)$. Hence, the next Corollary holds.

COROLLARY 3.7 *Let S be a sequence of full p-skyline relations*

$$\succ_1, \ldots, \succ_n \in \mathcal{F}_{\mathcal{H}}$$

such that for all $i \in [1, n-1]$, *we have that* $\succ_{i+1}$ *is a full p-skyline extension of* $\succ_i$. *Then* $|S| = O(|\mathcal{A}|^2)$.

3.7 p-skyline query evaluation

Here we show some methods of winnow query evaluation for p-skyline relations. We assume that we are given a full p-skyline relation $\succ$, a database relation r instance, and want to compute $w_{\succ}(r)$, i.e. all the best tuples in r according to $\succ$.

An important property of p-skyline relations is that they can be expressed using accumulation operators. Hence, a winnow query with a p-skyline relation can be represented as a Preference SQL query [KK02]. To do that, one needs to represent atomic preference relations using certain *base preference constructors*. Moreover, by construction, the size of a Preference SQL query based on p-skyline relation is linear in the number of attributes involved in the preference relation. A number of optimization techniques have been developed for such queries [HK05].

Evaluating of p-skyline winnow queries as Preference SQL queries requires access to a special query evaluation engine. If such an engine is not available, then general preference query evaluation techniques may be used. Some query optimization techniques in this area use preference-formula representation of a preference relation [Cho07a]. In Proposition 3.3, we showed that p-skyline relations can be represented as polynomial size preference formulas.

A number of preference query evaluation methods developed for skyline queries can also be applied to p-skyline winnow queries as well. One of them is BNL proposed in [BKS01]. This algorithm relies on the efficiency of dominance testing for skyline relations. As we showed here, dominance testing in the p-skyline framework can also be performed efficiently: using Theorem 3.5 or preference formula evaluation.

Another skyline algorithm called SFS [CGGL03] may also be adopted to evaluate p-skyline winnow queries. SFS uses the fact that for any SPO relation there exists a weak order relation containing it. In that algorithm, tuples of a relation instance r are ordered according to such a weak order, and then a version of BNL is run against the sorted r. As we show here, a weak order relation containing a given full p-skyline relation can be computed efficiently. By Theorem 3.4, a p-skyline relation $\succ_{ext}$ is a full p-skyline extension of another p-skyline relation $\succ$ if the edge set of $\Gamma_{\succ}$ is a subset of the edge set of $\Gamma_{\succ_{ext}}$. By Theorem 3.8, $\succ_{ext}$ is a weak order if its p-graph is a total order of $\mathcal{A}$. Moreover, since a total order is a maximal SPO, for any p-graph there is a total order p-graph containing it. Hence, the problem of computing a weak order p-skyline relation $\succ_{ext}$ containing $\succ$ can be reduced to the problem of computing a total order graph containing $\Gamma_{\succ}$. That can be done using topological sorting [CLRS01].

THEOREM 3.8 *Let $\succ$ be a full p-skyline relation such that the p-graph $\Gamma_{\succ}$ is a total order of $\mathcal{A}$. Then $\succ$ is a weak order.*

PROOF

By Corollary 3.1, $\succ$ can be represented as

$$\succ \equiv (((\succ_{A_1} \ \& \ \succ_{A_2}) \ \& \ \succ_{A_3}) \ldots) \ \& \ \succ_{A_n} .$$

Let us denote $\succ_{A_1}$ as $\succ_1$, and $\succ_i = \succ_{i-1} \ \& \ \succ_{A_i}$. Then $\succ = \succ_n$. We use induction to show that $\succ_n$ is a weak order. It is clear that a preference relation induced by an atomic preference relation is a weak order since all atomic preference relations we consider here are total orders. Hence, $\succ_1$ is a weak order. Assume that $\succ_{i-1}$ is a weak order. Thus, $\forall o_1, o_2, o_3 \ . \ o_1 \succ_{i-1} o_3 \Rightarrow o_1 \succ_{i-1} o_2 \lor o_2 \succ_{i-1} o_3$. The relation $\succ_i$ can be represented as

$$\succ_i = \succ_{i-1} \cup \approx_{Var(\succ_{i-1})} \cap \succ_{A_i} .$$

We prove that $\forall o_1, o_2, o_3 \ . \ o_1 \succ_i o_3 \Rightarrow o_1 \succ_i o_2 \lor o_2 \succ_i o_3$, i.e., $\succ_i$ is a weak order. Take any o_1, o_2, o_3 such that $o_1 \succ_i o_3$. If $o_1 \succ_{i-1} o_3$, then $o_1 \succ_{i-1} o_2 \lor o_2 \succ_{i-1} o_3$ since $\succ_{i-1}$ is a weak order. Hence, $o_1 \succ_i o_2 \lor o_2 \succ_i o_3$. Now let $(o_1, o_3) \in \approx_{Var(\succ_{i-1})} \cap \succ_{A_i}$. If (o_1, o_2) or (o_2, o_3) is in $\approx_{Var(\succ_{i-1})}$, then either (o_1, o_2) or (o_2, o_3) is in $\approx_{Var(\succ_{i-1})} \cap \succ_{A_i}$ because $\succ_{A_i}$ is a total order. If neither (o_1, o_2) nor (o_2, o_3) is in $\approx_{Var(\succ_{i-1})}$, then $o_1 \succ_{i-1} o_2$ or $o_2 \succ_{i-1} o_3$ by Theorem 3.5. Thus, $o_1 \succ_i o_2 \lor o_2 \succ_i o_3$, and $\succ_i$ is a weak order. □

3.8 Related work

The p-skyline framework is based on the preference constructor approach proposed by Kießling in [Kie02]. That framework was extended in [Kie05] by relaxing definitions of the accumulation operators and using *SV*-relations, instead of equality, as indifference relations. [Kie05] shows that such an extension preserves the SPO properties of accumulated relations. Moreover, the resulting accumulated relations were shown to be *larger* (in the set theoretic sense) than the relations composed using the equality-based accumulational operators. However, the problems of relative importance of attributes induced by such preference relations were not addressed in [Kie02] and [Kie05]. The problems of containment of preference relations and minimal extensions were also not considered.

The original skyline framework was proposed by Börzsönyi et al in [BKS01], where they proposed an extension of SQL in which the skyline operator can be intuitively represented. They also showed a number of algorithms for computing skylines: the block nested loop algorithm (BNL) and divide-and-conquer algorithm with some optimizations. Since then, a number of algorithms of computing skylines have been proposed. [TEO01] developed two algorithms – Bitmap and Index – to compute skyline points progressively. Another progressive skyline computation algorithm was proposed by Kossmann in [KRR02]. In the SFS algorithm proposed in [CGGL03], Chomicki et al showed that sorting a relation before computing a skyline over it using BNL gives computational benefits. In [CET05], Chan et al studied the problem of computing skylines with SPO atomic preferences. Some algorithms for progressive computation and maintenance of skylines were proposed by Lee in [LZLL07]. In [GSG07], Godfrey et al proposed the LESS skyline algorithm. It extends SFS by performing early elimination of tuples which are guaranteed not to be skyline tuples. To do that, [GSG07] proposed to use a window of tuples with high values of the *entropy score*. It was shown there that tuples with higher values of this score tend to dominate a large number of tuples. In that work, Godfrey et al performed a thorough analysis of running time of some widely used skyline algorithms.

In [GSG05] Godfrey at al showed that the number of skyline points in a dataset may be exponential in the number of attributes. Hence, if many attributes are relevant to user preferences, then the set of the best tuples according to such a preference relation may be rather large. A number of variants of the skyline operator have been proposed to solve the large-skyline-size problem. Chan et al [CJT$^+$06] propose to compute the set of *k-dominant* skyline points instead of the entire skyline. A point p is said to *k-dominate* another point q if there are k dimensions in which p is better than or equal to q, and in at least one dimension, p is better than q. A point that is not k-dominated by other points is in the *k-dominant skyline*. It was shown there that the existing skyline algorithms cannot be used to compute k-dominant skylines. Chan et al proposed several algorithms for such computations. Another variant of the skyline operator was presented in [LYZZ07]. It computes *k most representative* tuples of a skyline. k skyline points are called most representative if the number of points which are dominated by them is maximized. It was shown there that in the case of two attributes, the problem of computing the k most representative skyline tuples has an efficient polynomial solution. When the number of involved attributes is greater than two, the problem is NP-hard in general. For such cases, [LYZZ07] proposed a polynomial time approximation algorithm.

Another direction of research considered in the skyline framework is *subspace skyline* [PJET05, YLL$^+$05]. It is motivated by the fact that when a user provides a set of relevant attributes and the corresponding atomic preferences, he or she may decide later that some of them are not actually relevant and need not be considered. Hence there is a need in computing skyline not only over a given set of atomic preferences but also over its subsets. Such skylines are called *subspace skylines*. One interesting problem in this framework is to identify the subspaces such that a given tuple belongs to the corresponding skylines. In [PJET05], Pei et al showed an approach of computing such subspaces using the notion of *decisive subspace*. Another problem here is related to the computation of subspace skylines. It is clear that a subspace skyline may be computed using any skyline algorithm. However, to compute k subspace skylines (for k different subsets of relevant atomic preferences), such method may be inefficient since it does not use the information of possible similarity of the corresponding subspace skylines. To make the computation efficient, [PJET05] proposed an algorithm for computing *all* subspaces skylines at once. Another method computing all subspace skylines was proposed by Yuan et al in [YLL$^+$05]. It is based on the notion of *skyline cube*. It was shown there that if there are no two tuples in a dataset which share a value of the same attribute (this property is called the *distinct value property*), then the skyline of a smaller subspace is contained in the skyline of a larger subspace. Yuan et al [YLL$^+$05] used that property to compute a skyline cube – the set of all the subspace skylines for a given set of atomic preferences. The skyline cube approach was used by Lee et al in [LwYwH09] to find the *most interesting* subspaces given the maximum on the size of the corresponding skyline and a total order of attributes which represents the importance of attributes for a user.

We note that the approach of subspace skyline is in a sense orthogonal to the p-skyline framework proposed here. Both approaches extend the skyline framework. In the subspace skyline framework, the relative importance of attributes is fixed (i.e., all considered attributes are of equal importance) while the sets of the corresponding attributes may vary. In the p-skyline approach, the set of relevant attributes is fixed while the relative importance of them may vary. However, given a set of atomic preference relations, all subspace skylines and the results of all full p-skyline relations are subsets of the (full-space) skyline (under the distinct value property in the former case).

Chapter 4

Discovery of p-skyline relations

In Chapter 3, we proposed a class of preference relations called *p-skyline relations*. In this chapter, we introduce a method of constructing p-skyline relations based on user provided feedback.

4.1 Feedback based discovery of p-skyline relations

As we showed in the previous chapter, the p-skyline framework is a generalization of the skyline framework. The main difference between them is that in the p-skyline framework, we can deal with variable attribute importance. However, one of the main properties of the skyline framework is the simplicity of representing preferences. Namely, one needs to provide only a set of atomic preferences to construct a preference relation. For p-skylines, an additional piece of information – the relative importance of the attributes – has to be provided by a user. However, requiring users to describe attribute importance explicitly seems impractical for several reasons. First, the number of comparisons may be rather large: one has to compare all attributes pairwise. The second issue is even more serious – users themselves may be not fully aware of their own preferences.

In this chapter, we propose an alternative approach to discovering attribute importance relationships, based on *user feedback*. We use the following scenario of p-skyline relation discovery. A fixed set of tuples describing real objects is stored in a database relation $O \subseteq \mathcal{U}$. We assume that the user has preferences over these objects. Given O, she can partition it into disjoint subsets: the set G of tuples she confidently likes (*superior* examples), the set W of tuples she confidently dislikes (*inferior* examples), and the set of remaining tuples about which she is not sure. We assume here that if a tuple o is inferior, then there is at least one superior example which the user likes more than o. This assumption is justified by a general principle that we consider something bad because we know of a better alternative. Given that setup, we aim to compute a p-skyline relation according to which all tuples in G are superior and all tuples in W are inferior.

Recall that any p-skyline relation is defined by the following parameters:

- the set of relevant attributes;
- atomic preferences over the relevant attributes; and
- the combination of accumulation operators which are used to combine the atomic preferences into one relation.

Hence, the problem of discovery of a p-skyline relation based on user feedback can be decomposed into three problems of discovering each of the parameters above. We note that the first two problems have been studied in many frameworks of preference discovery. Some methods of solving them are shown in Section 4.6. Here we focus on the last problem: computing a p-skyline relation based on user feedback given that the set of relevant attribute is the entire set of attributes $\mathcal{A}$. In addition to that, we assume that atomic preference relations $\mathcal{H}$ are also known.

Having subsets G and W of a set of tuples O, we want to find a possible p-skyline relation which implies G being superior and W being inferior examples in O. Formally, we want to compute a full p-skyline relation $\succ$ (i.e., a member of $\mathcal{F}_{\mathcal{H}}$) such that 1) $G \subseteq w_{\succ}(O)$

(i.e. the tuples in G are among the most preferred tuples in O according to $\succ$), and 2) for every tuple o' in W, there is a tuple o in G such that $o \succ o'$ (i.e., o' is an inferior example). Such a p-skyline relation $\succ$ is called *favoring G and disfavoring W in O*. We may also refer to such p-skyline relations as *favoring/disfavoring* when the context is clear.

The first problem we consider is the existence of a p-skyline relation favoring G and disfavoring W in O.

Problem 1. (`DF-PSKYLINE`) *Given a set of atomic preference relations $\mathcal{H}$, a set of superior examples G and inferior examples W in a set O, determine if there exists a full p-skyline relation $\succ \in \mathcal{F}_{\mathcal{H}}$ favoring G and disfavoring W in O.*

In many real life scenarios, knowing that a favoring/disfavoring p-skyline relation exists is not sufficient. In addition to that, it is useful to know such a relation. We call this class of problems `FDF-PSKYLINE`. It is a *functional version* [Pap94] of `DF-PSKYLINE`. In particular, given subsets G and W of O, an instance of `FDF-PSKYLINE` outputs "no" if there is no $\succ \in \mathcal{F}_{\mathcal{H}}$ favoring G and disfavoring W in O. Otherwise, it outputs *some* full p-skyline relation favoring G and disfavoring W in O.

`EXAMPLE 4.1` *Let the set O consist of the following tuples describing cars on sale.*

id	*make*	*price*	*year*
t_1	*ford*	*30k*	*2007*
t_2	*bmw*	*45k*	*2008*
t_3	*kia*	*20k*	*2007*
t_4	*ford*	*40k*	*2008*
t_5	*bmw*	*50k*	*2006*

Assume also Mary wants to buy a car and her preferences over automobile attributes are as follows.

$>_{make}$: *BMW is better than Ford, Ford is better than Kia*

$>_{year}$: *higher values of year (i.e., newer cars) are preferred*

$>_{price}$: *lower values of price (i.e., cheaper cars) are preferred*

Let $G = \{t_4\}$, $W = \{t_3\}$. We discover a full p-skyline relation $\succ$ favoring G and disfavoring W. First, $>_{make}$ cannot be more important than all other attribute preferences since then t_2 and t_5 dominate t_4 and thus t_4 is not superior. Moreover, $>_{price}$ cannot be more important than the other attribute preferences because then t_3 and t_1 dominate t_4. However, if $>_{year}$ is more important than the other attribute preferences, then t_4 dominates t_3 and no other tuple dominates t_4 in $>_{year}$. At the same time, both t_2 and t_4 are the best according to $>_{year}$, but t_2 may dominate t_4 in $\succ_{make}$. Therefore, $>_{make}$ should not be more important than $>_{price}$. Thus, the following p-skyline relation favors G and disfavors W in O

$$\succ_1 = \succ_{year} \ \& \ (\succ_{price} \ \otimes \ \succ_{make})$$

The set of the best tuples in O according to $\succ_3$ is $\{t_2, t_4\}$.

Generally there may be zero or more full p-skyline relations favoring G and disfavoring W in O. When more than one such preference relation exists, we pick the largest ones (in the set theoretic sense). Larger preference relations imply more dominated tuples and fewer most preferred ones. Consequently, the result of $w_{\succ}(O)$ gets more manageable due to its decreasing size. Moreover, maximizing $\succ$ corresponds to minimizing $w_{\succ}(O) - G$, which implies more precise correspondence of $\succ$ to the real user preferences. A maximal full p-skyline relation favoring G and disfavoring W in O is called *optimal*.

Thus, the second problem considered here is computing an optimal p-skyline relation favoring G and disfavoring W.

Problem 2. (`OPTIMAL FDF-PSKYLINE`) *Given a set of atomic preference relations $\mathcal{H}$, a set of superior examples G and inferior examples W of a set O, compute an optimal p-skyline relation $\succ \in \mathcal{F}_{\mathcal{H}}$ favoring G and disfavoring W in O.*

EXAMPLE 4.2 *Take G, W, and $\succ_1$ from Example 4.1. Note that in order to make t_4 dominate t_2, we need to make price more important than year. As a result, the relation*

$$\succ_2 = \succ_{year} \ \& \ \succ_{price} \ \& \ \succ_{make}$$

also favors G and disfavors W in O but the set of best tuples in O according to $\succ_2$ is $\{t_4\}$. Moreover, $\succ_2$ is optimal. *The justification is that no other p-skyline relation favoring G and disfavoring W contains $\succ_2$ since the p-graph of $\succ_2$ is a total order of the attributes $\{year, price, make\}$ and thus is a maximal SPO.*

Even though the notion of optimal favoring/disfavoring reduces the space of alternative p-skyline relations, there still may be more than one optimal favoring/disfavoring p-skyline relation, given G, W, and O.

4.2 Constraints to discover p-skyline relations

In this section, we show constraints that apply to the p-graph of a p-skyline relation and that guarantee that the corresponding p-skyline relation favors G and disfavors W in O.

Let us consider the notion of favoring G in O first. For any member $o' \in G$ to be in the set of the most preferred tuples of O, o' must not be dominated by any tuple in O. That is,

$$\forall o \in O, o' \in G \ . \ o \not\succ o' \tag{4.1}$$

Using Theorem 3.5, we can rewrite (4.1) as

$$\forall o \in O, o' \in G \ . \ Ch_{\Gamma_\succ}(BetterIn(o,o')) \not\supseteq BetterIn(o',o), \tag{4.2}$$

for $BetterIn(o_1,o_2) = \{A \mid o_1.A >_A o_2.A\}$. Note that no tuple can be preferred to itself by irreflexivity of $\succ$. Therefore, a p-skyline relation favoring G in O should satisfy $(|O|-1)\cdot|G|$ *negative* constraints τ in the form:

$$\tau : Ch_{\Gamma_\succ}(\mathcal{L}_\tau) \not\supseteq \mathcal{R}_\tau$$

where $\mathcal{L}_\tau = BetterIn(o,o'), \mathcal{R}_\tau = BetterIn(o',o)$. We denote this set of constraints as $\mathcal{N}(G,O)$.

EXAMPLE 4.3 *Take Example 4.1. Then any p-skyline relation $\succ \in \mathcal{F}_\mathcal{H}$ favoring $G = \{t_3\}$ in O has to satisfy each negative constraint below*

$t_1 \not\succ t_3$	$Ch_{\Gamma_\succ}(\{make\}) \not\supseteq \{price\}$
$t_2 \not\succ t_3$	$Ch_{\Gamma_\succ}(\{make, year\}) \not\supseteq \{price\}$
$t_4 \not\succ t_3$	$Ch_{\Gamma_\succ}(\{make, year\}) \not\supseteq \{price\}$
$t_5 \not\succ t_3$	$Ch_{\Gamma_\succ}(\{make\}) \not\supseteq \{price, year\}$

Now consider the notion of disfavoring W in O. According to the definition, a p-skyline relation $\succ$ disfavors W in O if and only if the following holds

$$\forall o' \in W\ \exists o \in G\ .\ o \succ o'. \tag{4.3}$$

Following Theorem 3.5, it can be rewritten as a set of *positive constraints $\mathcal{P}(W,G)$*

$$\forall o' \in W \bigvee_{o_i \in G} Ch_{\Gamma_\succ}(BetterIn(o_i,o')) \supseteq BetterIn(o',o_i). \tag{4.4}$$

Therefore, in order for $\succ$ to disfavor W in O, its p-graph $\Gamma_\succ$ has to satisfy $|W|$ positive constraints.

EXAMPLE 4.4 *Take Example 4.1. Then any p-skyline relation $\succ \in \mathcal{F}_\mathcal{H}$ favoring $G =$*

$\{t_1, t_3\}$ *and disfavoring* $W = \{t_4\}$ *in* O *has to satisfy the constraint*

$$t_1 \succ t_4 \vee t_3 \succ t_4$$

which is equivalent to the following positive constraint

$$Ch_{\Gamma_\succ}(\{price\}) \supseteq \{year\} \vee Ch_{\Gamma_\succ}(\{price\}) \supseteq \{year, make\}$$

Let us summarize the constraints we have considered here. In order to construct a p-skyline relation $\succ$ favoring G and disfavoring W in O, we need to construct an attribute importance graph $\Gamma_\succ$ which satisfies `SPO+Envelope` to guarantee that $\succ$ is a p-skyline relation, $\mathcal{N}(G, O)$ to guarantee favoring G in O, and $\mathcal{P}(W, G)$ to guarantee disfavoring W in O.

By Theorem 3.4, a p-graph of an optimal $\succ$ is *maximal* among all graphs which satisfy `SPO+Envelope`, $\mathcal{N}(G, O)$, and $\mathcal{P}(W, G)$.

4.3 Using superior/inferior examples for p-skyline discovery

In this section, we study the problems of existence of a favoring/disfavoring p-skyline relation and computing a favoring/disfavoring p-skyline relation. We start with the problem existence of such a relation.

`THEOREM 4.1 DF-PSKYLINE` *is NP-complete.*

`PROOF`

The favoring/disfavoring p-skyline existence problem is in NP since checking if a p-skyline relation $\succ$ favors G and disfavors W in O can be done in polynomial time by evaluating

$w_{\succ}(O)$, checking $G \subseteq w_{\succ}(O)$, and checking if for every member of W there is a member of W dominating it.

In order to show the hardness result, we do a polynomial-time reduction from SAT. This is a two-step reduction. First, we show that for every instance ϕ of SAT there are corresponding instances of positive $\mathcal{P}$ and negative $\mathcal{N}$ constraints, and ϕ has a solution iff $\mathcal{P}$ and $\mathcal{N}$ are satisfiable. Second, we show that for every such $\mathcal{P}$ and $\mathcal{N}$ there are corresponding instances of G, W, and O.

Consider instances of SAT in the following form

$$\phi(x_1,\ldots,x_n) = \psi_1(x_1,\ldots,x_n) \wedge \ldots \wedge \psi_m(x_1,\ldots,x_n)$$

where

$$\psi_t(x_1,\ldots,x_n) = \widehat{x_{i_t}} \vee \ldots \vee \widehat{x_{j_t}}$$

For every instance of ϕ, construct $\mathcal{A} = \{c, y_1, \overline{y_1}, y'_1, \ldots, y_n, \overline{y_n}, y'_n\}$. The sets of positive and negative constraints are constructed as follows. For every variable x_i,

1. Create positive constraints

$$\begin{aligned} \chi_i &: (y_i, c) \in \Gamma \vee (\overline{y_i}, c) \in \Gamma \\ \pi_i &: (\overline{y_i}, y'_i) \in \Gamma \end{aligned}$$

2. Create negative constraints

$$\begin{aligned} \lambda_i^1 &: (\overline{y_i}, y_i) \notin \Gamma \\ \lambda_i^2 &: (y_i, y'_i) \notin \Gamma \\ \lambda_i^3 &: (y'_i, c) \notin \Gamma \end{aligned}$$

Now, for every $\psi_t(x_1,\ldots,x_n) = \widehat{x_{i_t}} \vee \ldots \vee \widehat{x_{j_t}}$ of ϕ construct the following positive constraint

$$\mu_t : (\widehat{y_{i_t}}, c) \in \Gamma \vee \ldots \vee (\widehat{y_{i_t}}, c) \in \Gamma$$

where $\widehat{y_i} = \begin{cases} y_i & \text{if } \widehat{x_i} = x_i \\ \overline{y_i} & \text{if } \widehat{x_i} = \overline{x_i} \end{cases}$.

We claim that there is a satisfying assignment $(v_i,\ldots,v_n)$ for ϕ iff there is a p-graph satisfying all the constraints above. First, assume there is a p-graph Γ satisfying all the constraints above. Construct the assignment $v = (v_i,\ldots,v_n)$ as follows:

$$v_i = \begin{cases} 0 & \text{if } (\overline{y_i}, c) \in \Gamma \\ 1 & \text{if } (y_i, c) \in \Gamma \end{cases}.$$

Since $\Gamma_\succ$ satisfies all χ_i, for every i we have $(y_i,c) \in \Gamma$ or $(\overline{y_i},c) \in \Gamma$. Thus, every v_i will be assigned to some value according to the rule above. Now prove that v_i is assigned to only one value, i.e., we cannot have both $(y_i,c) \in \Gamma$ and $(\overline{y_i},c) \in \Gamma$. Since Γ satisfies π_i, we have $(\overline{y_i}, y_i') \in \Gamma$. Thus having both $(y_i,c) \in \Gamma$ and $(\overline{y_i},c) \in \Gamma$ and `Envelope` implies

$$(\overline{y_i}, y_i) \in \Gamma \vee (y_i, y_i') \in \Gamma \vee (y_i', c) \in \Gamma$$

However, the expression above violates the constraints $\lambda_i^1, \lambda_i^2, \lambda_i^3$. Therefore, exactly one of $(y_i,c) \in \Gamma$, $(\overline{y_i},c) \in \Gamma$ holds.

Take every μ_t. Since it is satisfied by Γ, the corresponding ψ_i must be also satisfied by construction of μ_t. Therefore, ϕ is also satisfied.

Now assume that there is an assignment $(v_1,\ldots,v_n)$ satisfying ϕ. Show that there is a p-graph $\Gamma_\succ$ satisfying all the constraints above. Here we construct such a graph.

For every $i \in [1,n]$, draw the edge

$$(y_i, c) \in \Gamma_{\succ} \quad \text{if } v_i = 1\text{, and} \tag{P1}$$

$$(\overline{y_i}, c) \in \Gamma_{\succ}, \quad \text{otherwise} \tag{P2}$$

This satisfies the constraint χ_i. Moreover, all the constraints μ_t are satisfied by construction. Now, for every $i \in [1,n]$, draw the edge

$$(\overline{y_i}, y'_i) \in \Gamma_{\succ} \tag{P3}$$

which satisfies the constraint π_i. As a result, all positive constraints are satisfied. Moreover, none of the edges above violates any negative constraint. Thus, all the constraints above are satisfied.

In additional the edges above, let us draw the following edges

1. for every i, j such that $v_i = 0, v_j = 0$, draw the edge

$$(\overline{y_i}, y'_j) \in \Gamma_{\succ} \tag{P4}$$

   It is clear that these edges do not violate any negative constraints above.

2. for every i, j such that $v_i = 0, v_j = 1$, draw the edge

$$(\overline{y_i}, y_j) \in \Gamma_{\succ} \tag{P5}$$

   Since $i \neq j$, this edge does not violate any negative constraints above.

It is easy to verify that the constructed graph $\Gamma_{\succ}$ satisfies `SPO+Envelope` and all the negative and positive constraints above.

Now let us show that there exist sets of objects O, G and W which can be used to obtain the constraints $\chi_i, \pi_i, \lambda_i^1, \lambda_i^2, \lambda_i^3, \mu_t$. Let for every attribute in $\mathcal{A}$, its domain contain at least three numbers $\{-1, 0, 1\}$, and greater values of be preferred. Here we construct the sets G, W, M, and $O = G \cup W \cup M$ that generate the positive and negative constraints above.

1. Let G consist of a single object g with all attributes values are 0.

2. Let $W = \{b_1, \ldots, b_n, u_1, \ldots, u_n, w_1, \ldots, w_m\}$ be constructed as follows:

   - for every $i \in [1, \ldots, n]$, let all the attributes of b_i be equal to 0 except for the value of $\overline{y_i}$ be -1 and y_i' be 1.
   - for every $i \in [1, \ldots, n]$, let all the attributes of u_i be equal to 0 except for the values of $y_i, \overline{y_i}$ be equal to -1 and the value of c be equal to 1.
   - take every $t \in [1, \ldots, m]$ and let $\mu_t : (\widehat{y_{i_t}}, c) \in \Gamma \vee \ldots \vee (\widehat{y_{j_t}}, c) \in \Gamma$, where $\widehat{y_i} \in \{y_i, \overline{y_i}\}$. Let all attributes of w_t be equal to 0 except for the value of c be 1, and the values of $\widehat{y_{i_t}}, \ldots, \widehat{y_{j_t}}$ (whatever they are) be -1.

3. Let $M = \{m_1^1, m_1^2, m_1^3, \ldots, m_n^1, m_n^2, m_n^3\}$ be constructed as follows. For all $i \in [1, \ldots, n]$,

   - Let all attributes of m_i^1 be 0 except for y_i and $\overline{y_i}$ be -1 and 1 respectively.
   - Let all attributes of m_i^2 be 0 except for y_i and y_i'be 1 and -1 respectively.
   - Let all attributes of m_i^3 are 0 except for y_i' and c be 1 and -1 respectively.

It can be easily shown that these sets of objects induce the set of constructed constraints.
□

Now consider the problems of computing favoring/disfavoring p-skyline relations. First, we consider the problem of computing any p-skyline relation favoring G and disfavoring W in O, not necessary minimal. After that, we address the problem of computing optimal p-skyline relations. The results shown below are based on the following lemma.

LEMMA 4.1 *Take a full p-skyline relation $\succ$, and subsets G and W of O. Then the next two operations can be done in polynomial time:*

1. *verifying if $\succ$ is optimal favoring G and disfavoring W in O;*
2. *computing an optimal p-skyline relation $\succ_{ext}$ favoring G and disfavoring W in O which is a full p-skyline extension of $\succ$.*

PROOF

See Appendix A.

THEOREM 4.2 FDF-PSKYLINE *is* FNP-*complete*

PROOF

Given two disjoint subsets G and W of O and $\succ \in \mathcal{F}_{\mathcal{H}}$, checking if $\succ$ favors G and disfavors W in O can be done in polynomial time. In particular, one needs to compute $w_{\succ}(O)$, check $G \subseteq w_{\succ}(O)$ and verify if for every $o \in W$ there is $o' \in G$ such that $o' \succ o$. Hence, FDF-PSKYLINE is in FNP.

Now show that FDF-PSKYLINE is FNP-hard. To do that, we use a reduction from FSAT. In particular, we find functions R and S, both computable in logarithmic space, such that 1) for any instance x of FSAT, $R(x)$ is an instance of FDF-PSKYLINE, and 2) for any correct output z of $R(x)$, $S(z)$ is a correct output of x. For such a reduction, we use the construction from the proof of Theorem 4.1. There we showed how a relation (denote it as $\succ$) satisfying all the constraints (and thus favoring the constructed G and W) may be obtained. In the current reduction, if there is a p-skyline relation favoring G and disfavoring W in O, then the relation $\succ$ itself is returned. Otherwise, "no" is returned.

The function R mentioned above has to convert an instance of FSAT to an instance of FDF-PSKYLINE (i.e., G, W, and O). In the reduction shown in the proof of Theorem 4.1, such a transformation is done via a set of constraints. However, it is easy to observe that such a construction can be performed using the corresponding instance of `FSAT`. By the construction, the sets G, M, and the subset $\{b_1,\ldots,b_n,u_1,\ldots,u_n\}$ of W are common for any instance of `FSAT` with n variables. To construct the subset $\{w_1,\ldots,w_m\}$ of W, one can use the expression ψ_t instead of the corresponding constraint μ_t. It is clear that the function R performing such a transformation can be evaluated in logarithmic space.

We construct the function S as follows. If the instance of `FDF-PSKYLINE` returns "no", S returns "no". Otherwise, it constructs the satisfying assignment $(v_1,\ldots,v_n)$ in the following way: for every i, v_i is set to 1 if the p-graph contains the edge $(y_i,c) \in \Gamma_{\succ}$, and 0 otherwise. It is clear that such a computation may be done in logarithmic space. $\square$

Surprisingly, the problem of computing an optimal favoring/disfavoring p-skyline relation is not harder then the problem of computing any favoring/disfavoring p-skyline relation.

THEOREM 4.3 OPTIMAL FDF-PSKYLINE *is FNP-complete*

PROOF
Given $\succ \in \mathcal{F}_{\mathcal{H}}$, checking if it is optimal favoring G and disfavoring W can be done in polynomial time (Lemma 4.1). Hence, `OPTIMAL FDF-PSKYLINE` is in `FNP`.

To show that it is `FNP`-hard, we reduce from `FDF-PSKYLINE`. Here we construct the function F that takes a p-skyline relation or "no" and returns a p-skyline relation or "no". F returns "no" if its input is "no". If its input is a p-skyline relation $\succ$, it returns an optimal extension of $\succ$ as shown in Lemma 4.1. As a result, F returns an optimal favoring/disfavoring p-skyline relation iff the corresponding favoring/disfavoring p-skyline relation exists. The functions R and S transforming inputs of `FDF-PSKYLINE` to inputs of `OPTIMAL FDF-PSKYLINE` and outputs of `OPTIMAL FDF-PSKYLINE` to outputs of `FDF-PSKYLINE` correspondingly are trivial and hence are computable in logspace. Therefore, `FDF-PSKYLINE` is `FNP`-complete. $\square$

In view of Theorems 4.1, 4.2, and 4.3, we consider restricted versions of the favoring/disfavoring p-skyline relation problems. We will assume that there are no inferior examples ($W = \emptyset$).

4.4 Using only superior examples for p-skyline discovery

By definition of the problem of discovering favoring/disfavoring p-skyline relations, a user is required to identify superior as well as inferior examples in a set of tuples. In many real life scenarios, users are only indirectly involved in the process of identifying such classes. For instance, the click-through rate may be used to measure the popularity of products. Using this metric, it is easy to find the superior examples – the tuples with the highest click-through rate. However, the problem of identifying inferior examples – those which the user confidently dislikes – is harder. Namely, low click-through rate may mean that a tuple is inferior, as well as that the user does not know about it, or it simply does not satisfy the search criteria. Thus, there is a need for discovery of p-skyline relations based on superior examples only.

Let us denote the subclasses of `DF-PSKYLINE`, `FDF-PSKYLINE`, and `OPTIMAL FDF-PSKYLINE` in which the sets of inferior examples W are empty as `F-PSKYLINE`, `FF-PSKYLINE`, and `OPTIMAL FF-PSKYLINE`, correspondingly.

As we show further, these problems are simpler than the corresponding problems for favoring/disfavoring p-skyline relations. In particular, we show polynomial time algorithms for all of them.

Consider `F-PSKYLINE` first. In Corollary 3.4 we showed that the set of the best objects according to the skyline preference relation is the largest among all p-skyline relations. Hence, the next proposition holds.

`PROPOSITION 4.1` *There exists a full p-skyline relation $\succ \in \mathcal{F}_{\mathcal{H}}$ favoring G in O if and* *only if*

$$G \subseteq w_{\mathrm{sky}_{\mathcal{H}}}(O).$$

Proposition 4.1 implies that to solve F-PSKYLINE, one needs to run a skyline algorithm over O and check if the result contains G. This clearly can be done in polynomial time. Moreover, FF-PSKYLINE can also be solved in polynomial time: if $G \subseteq w_{sky_{\mathcal{H}}}(O)$, then relation $sky_{\mathcal{H}}$ is a relation favoring G and disfavoring W in O. Otherwise, there is no such a relation.

Now consider OPTIMAL FF-PSKYLINE. To construct a full p-skyline relation $\succ$ favoring G in O, we need to construct the corresponding graph $\Gamma_{\succ}$ which satisfies $\mathcal{N}(G,O)$ and SPO+Envelope. Furthermore, to make the relation $\succ$ optimal favoring G in O, $\Gamma_{\succ}$ has to be a *maximal* graph satisfying these constraints. In the next section, we present an algorithm for constructing optimal p-skyline relations.

4.4.1 p-skyline syntax tree transformation

The approach of constructing optimal favoring p-skyline relations we propose here is based on iterative transformations of normalized p-skyline syntax trees. We assume that the provided set of superior examples G satisfies Proposition 4.1, i.e., $G \subseteq w_{sky_{\mathcal{H}}}(O)$. The idea beyond our approach is as follows. First, we generate the set of negative constraints $\mathcal{N}(G,O)$. The p-skyline relation we start with is $sky_{\mathcal{H}}$ since it is the least full p-skyline relation favoring G in O. In every iteration of the algorithm, we pick an atomic preference relation in $\mathcal{H}$ and apply to the current p-skyline relation's syntax tree a fixed set of transformation rules. As a result, we obtain a "locally maximal" p-skyline relation satisfying *the given set $\mathcal{N}(G,O)$ of negative constraints*. Recall that a negative constraint in $\mathcal{N}(G,O)$ represents the requirement that a superior example from G is not dominated by a tuple in O. Eventually, this technique produces a maximal p-skyline relation satisfying $\mathcal{N}(G,O)$.

Let us describe what we mean by "locally maximal".

DEFINITION 4.1 *Let M be a nonempty subset of $\mathcal{A}$. A full p-skyline relation $\succ \in \mathcal{F}_{\mathcal{H}}$ which favors G in O such that $\Gamma_{\succ} \subset M \times M$ is M-*favoring G in O. *A maximal relation among all of them is called* optimal.

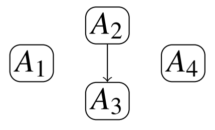
We note that similarly to an optimal favoring p-skyline relation, an optimal M-favoring relation is not unique for given G, O, and M.

id	A_1	A_2	A_3	A_4
t_1	0	0	0	0
t_2	1	0	-1	0
t_3	-1	1	-1	0
t_4	1	0	1	-1

(a) Set of tuples O

	$\mathcal{L}$	$\mathcal{R}$
τ_1	$\{A_1\}$	$\{A_3\}$
τ_2	$\{A_2\}$	$\{A_1,A_3\}$
τ_3	$\{A_1,A_3\}$	$\{A_4\}$

(b) Negative constraints $\mathcal{N}(G,O)$



(c) optimal M-favoring p-skyline relation

Figure 4-1: *Example 4.5*

EXAMPLE 4.5 *Let $\mathcal{A} = \{A_1,A_2,A_3,A_4\}$, $\mathcal{H} = \{>_{A_1},>_{A_2},>_{A_3},>_{A_4}\}$, where a greater value of the corresponding attribute is preferred according to every $>_{A_i}$. Let the set of objects O be as shown in Figure 4-1(a) and $G = \{t_1\}$. Then the set of negative constraints $\mathcal{N}(G,O)$ is shown in Figure 4-1(b). Consider the p-skyline relation $\succ$ represented by the p-graph $\Gamma_{\succ}$ shown in Figure 4-1(c). It is an optimal $\{A_1,A_2,A_3\}$-favoring relation: first, $\Gamma_{\succ}$ satisfies all the constraints in $\mathcal{N}(G,O)$; second, any additional edge from one attribute to another attribute in $\{A_1,A_2,A_3\}$ violates $\mathcal{N}(G,O)$. In particular, the edge $A_1 \rightarrow A_3$ violates τ_1 and the edge $A_2 \rightarrow A_1$ violates τ_2. Any other edge between $\{A_1,A_2,A_3\}$ induces one of the two edges above.*

At the same time, $\succ$ is not an optimal $\mathcal{A}$-favoring relation because the edge $A_4 \rightarrow A_1$ may be added to $\Gamma_{\succ}$ without violating $\mathcal{N}(G,O)$.

By Definition 4.1, the edge set of a p-graph of every optimal M-favoring relation is maximal among all the p-graphs of M-favoring relations. Note that if M is a singleton, the edge set of a p-graph $\Gamma_{\succ}$ of an optimal M-favoring relation $\succ$ is empty, i.e., $\succ = sky_{\mathcal{H}}$. If $M = \mathcal{A}$, then an optimal p-skyline relation M-favoring G in O is also an optimal p-skyline relation favoring G in O. Thus, if we had a method of transforming an optimal M-favoring to an optimal $(M \cup \{A\})$-favoring p-skyline relation for any attribute A, we could construct an optimal favoring p-skyline relation by induction. A useful property of such a transformation process is shown in the next lemma.

`LEMMA 4.2` *Let a full p-skyline relation $\succ \in \mathcal{F}_{\mathcal{H}}$ be an optimal M-favoring relation, and a full p-skyline extension $\succ_{ext}$ of $\succ$ be $(M \cup \{A\})$-favoring. Then any edge in $\Gamma_{\succ_{ext}} - \Gamma_{\succ}$ starts or ends in A.*

`PROOF`

Take $\Gamma_{\succ_{ext}}$ and construct Γ' from it by removing all edges going from and to A. Clearly, Γ' is an SPO. Now consider the `Envelope` property. Pick any four nodes of $\Gamma_{\succ}$ different from A. Since $\Gamma_{\succ_{ext}}$ is a p-graph, the `Envelope` property holds for the graph induced by these four nodes in $\Gamma_{\succ_{ext}}$. Assume it does not hold for the subgraph of Γ' induced by these nodes. That means that some edge between these four nodes is not present in Γ' which means that some of these nodes was A, i.e., we get a contradiction. Thus, Γ' satisfies the `Envelope` property as well, i.e., it's a p-graph of a p-skyline relation $\succ'$. Moreover, $\Gamma_{\succ} \subseteq \Gamma_{\succ'}$ since $\Gamma_{\succ}$ has no edges from/to A and $\Gamma_{\succ} \subseteq \Gamma_{\succ_{ext}}$. Since $\succ$ is optimal M-favoring, $\Gamma_{\succ} = \Gamma'$. Therefore, all edges in $\Gamma_{\succ_{ext}} - \Gamma_{\succ}$ go from or to A. $\square$

`EXAMPLE 4.6` *Consider $\mathcal{N}(G,O)$ from Example 4.5 also shown in Figure 4-2(a), and the optimal $\{A_1, A_2, A_3\}$-favoring relation $\succ$. Several different optimal $\mathcal{A}$-favoring p-skyline relations which contain $\succ$ are possible here. Two of them are $\succ_1$ and $\succ_2$ whose p-graphs are shown in Figures 4-2(b) and 4-2(c).*

In section 3.6, we showed four syntax tree transformation rules which can be used to extend a p-skyline relations in a minimal way. Although an optimal $(M \cup \{A\})$-favoring

	$\mathcal{L}$	$\mathcal{R}$
τ_1	$\{A_1\}$	$\{A_3\}$
τ_2	$\{A_2\}$	$\{A_1, A_3\}$
τ_3	$\{A_1, A_3\}$	$\{A_4\}$

(a) Negative constraints $\mathcal{N}(G, O)$

$$\begin{tikzcd}
 & \boxed{A_2} \arrow[d] \arrow[r] & \boxed{A_4} \arrow[dl] \\
\boxed{A_1} & \boxed{A_3} &
\end{tikzcd}$$

(b) $\Gamma_{\succ_1}$

$$\begin{tikzcd}
\boxed{A_4} \arrow[d] & \boxed{A_2} \arrow[d] \\
\boxed{A_1} & \boxed{A_3}
\end{tikzcd}$$

(c) $\Gamma_{\succ_2}$

Figure 4-2: *Example 4.6*

p-skyline relation is an extension of an optimal M-favoring p-skyline relation, it is not necessary a minimal extension in general. However, an important property of that set of rules is its completeness, i.e., any minimal extension can be computed using them. Hence, an optimal $(M \cup \{A\})$-favoring p-skyline relation can be produced from an optimal M-favoring p-skyline relation by *iterative application of the minimal extension rules*.

We use the following idea for constructing optimal $(M \cup \{A\})$-favoring relations. We start with an optimal M-favoring p-skyline relation $\succ_0$ and apply the transformation rules to $T_{\succ_0}$ in every possible way that guarantee that the new edges in the p-graph go only from or to A. In other words, we compute all minimal $(M \cup \{A\})$-favoring extensions of $\succ_0$. We compute such extensions until we find the first one which does not violate $\mathcal{N}(G, O)$. When we find it (denote it as $\succ_1$), we repeat all the steps above but for $\succ_1$. This process continues until for some $\succ_m$, every its minimal extension we compute as above violates $\mathcal{N}(G, O)$. Since in every iteration we compute all minimal $(M \cup \{A\})$-favoring extensions, $\succ_m$ is an optimal $(M \cup \{A\})$-favoring extension of $\succ_0$.

An important condition to apply Theorem 3.7 is that the input syntax tree for every transformation rule has to be normalized. At the same time, syntax trees returned by the transformation rules are not guaranteed to be normalized. Therefore, we need to normalize a tree before applying transformation rules to it.

Let us select the rules which can be used to construct an $(M \cup \{A\})$-favoring from an M-

favoring p-skyline relations. By Lemma 4.2, such rules may add *only* such edges to p-graph that go to A or from A. According to Observation 3.1, $Rule_1$ adds edges going to the node A if $C_{i+1} = A$ or $N_1 = A$. Similarly, $Rule_2$ adds edges going from A if $C_{i+1} = A$ or $N_m = A$. $Rule_3$ adds edges going from or to A if $C_i = A$ or $C_{i+1} = A$ correspondingly. However, $Rule_4$ can only be applied to a pair of & -nodes. Hence, as we showed in section 3.6, that rule adds edges going from at least two nodes to at least two different nodes of a p-graph. Hence, any application of $Rule_4$ violates Lemma 4.2. We conclude that $Rule_1, Rule_2, Rule_3$ *are sufficient to construct any optimal* $(M \cup \{A\})$*-favoring p-skyline relation.*

4.4.2 Efficient constraint checking

Before going to details of the algorithm of p-skyline relation discovery we outlined in the previous section, we consider an important step of the algorithm – testing if an extension of a p-skyline relation satisfies a set negative constraints. In this section, we propose an efficient method to do that.

Recall that a negative constraint is

$$\tau : Ch_{\Gamma_{\succ}}(\mathcal{L}_\tau) \not\supseteq \mathcal{R}_\tau$$

It can be visualized as two layers of nodes $\mathcal{L}_\tau$ and $\mathcal{R}_\tau$. For any full p-skyline relation $\succ \in \mathcal{F}_{\mathcal{H}}$ satisfying τ, its p-graph $\Gamma_{\succ}$ may induce edges going between the nodes of the layers $\mathcal{L}_\tau$ and $\mathcal{R}_\tau$. However, in order for $\succ$ to satisfy τ, there should be at least one member of $\mathcal{R}_\tau$ with no incoming edges from $\mathcal{L}_\tau$.

The method of efficient checking negative constraints against a p-graph we propose here is based on the fact that the edge set of the p-graph of a transformed p-skyline relation monotonically increases. Therefore, while we transform a p-skyline relation $\succ$, we can simply drop the elements of $\mathcal{R}_\tau$ which already have incoming edges from $\mathcal{L}_\tau$. If we do so after every transformation of the p-skyline relation $\succ$, the negative constraint τ will be violated by a $\Gamma_{\succ}$ only if $\mathcal{R}_\tau$ is empty. The next proposition says that such a modification of negative constraints is valid.

PROPOSITION 4.2 *Let a full p-skyline relation $\succ \in \mathcal{F}_{\mathcal{H}}$ satisfy a system of negative constraints $\mathcal{N}$. Construct the system of negative constraints $\mathcal{N}'$ from $\mathcal{N}$ in which every $\tau' \in \mathcal{N}'$ is created from a member τ of $\mathcal{N}$ in the following way*

- $\mathcal{L}_{\tau'} = \mathcal{L}_\tau$
- $\mathcal{R}_{\tau'} = \mathcal{R}_\tau - \{Y \in \mathcal{R}_\tau \mid \exists X \in \mathcal{L}_\tau \,.\, (X,Y) \in \Gamma_{\succ}\}$

Then any full p-skyline extension $\succ'$ of $\succ$ satisfies $\mathcal{N}$ if and only if $\succ'$ satisfies $\mathcal{N}'$.

A constraint τ' constructed from τ as shown in Proposition 4.2 is called a *minimal negative constraints w.r.t. $\succ$*. The corresponding system of negative constraints $\mathcal{N}'$ is called a *minimal system of negative constraints w.r.t. $\succ$*.

PROOF OF PROPOSITION 4.2.
$\boxed{\Leftarrow}$ Take any τ' from $\mathcal{N}'$ with the corresponding $\tau \in \mathcal{N}$. By construction, $\mathcal{L}_\tau = \mathcal{L}_\tau, \mathcal{R}_{\tau'} \subseteq \mathcal{R}_\tau$. Now assume $\succ'$ satisfies τ'. This means that

$$\exists B \in \mathcal{R}_{\tau'} \; \forall A \in \mathcal{L}_{\tau'} \; : \; (A,B) \notin \Gamma_{\succ'} \tag{4.5}$$

Now recall that $\mathcal{R}_{\tau'} \subseteq \mathcal{R}_\tau$. Thus $B \in \mathcal{R}_\tau$. This together with $\mathcal{L}_\tau = \mathcal{L}_{\tau'}$ and (4.5) gives

$$\exists B \in \mathcal{R}_\tau \; \forall A \in \mathcal{L}_\tau \,.\, (A,B) \notin \Gamma_{\succ'},$$

i.e., $\Gamma_{\succ'}$ satisfies τ.
$\boxed{\Rightarrow}$ Now let $\succ'$ satisfy τ. This means

$$\exists B \in \mathcal{R}_\tau \; \forall A \in \mathcal{L}_\tau \,.\, (A,B) \notin \Gamma_{\succ'} \tag{4.6}$$

Since $\succ \subseteq \succ'$, $\Gamma_{\succ} \subseteq \Gamma_{\succ'}$. Thus, if there is no edge from $\mathcal{L}_{\tau}$ to B in $\Gamma_{\succ'}$, then there is no such edge in its subset $\Gamma_{\succ}$. Recall that τ' is a *minimized* version of τ w.r.t. $\succ$. Thus, the lack of edge from $\mathcal{L}_{\tau}$ to B in $\Gamma_{\succ}$ implies $B \in \mathcal{R}_{\tau}$. This together with $\mathcal{L}_{\tau} = \mathcal{L}_{\tau'}$ and (4.6) gives

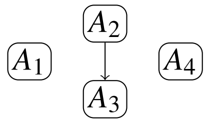
$$\exists B \in \mathcal{R}_{\tau'}\ \forall A \in \mathcal{L}_{\tau'}\ .\ (A,B) \notin \Gamma_{\succ'},$$

i.e., $\Gamma_{\succ'}$ satisfies τ'. $\square$

Minimization of a system of negative constraints is illustrated in the next example.

	$\mathcal{L}$	$\mathcal{R}$
τ_1	$\{A_1\}$	$\{A_3\}$
τ_2	$\{A_2\}$	$\{A_1,A_3\}$
τ_3	$\{A_1,A_3\}$	$\{A_4\}$

(a) Original system of negative constraints $\mathcal{N}$



$$\begin{tikzcd} & A_2 \arrow[d] & \\ A_1 & A_3 & A_4 \end{tikzcd}$$

(b) Optimal M-favoring p-skyline relation

	$\mathcal{L}$	$\mathcal{R}$
τ'_1	$\{A_1\}$	$\{A_3\}$
τ'_2	$\{A_2\}$	$\{A_1\}$
τ'_3	$\{A_1,A_3\}$	$\{A_4\}$

(c) Minimal system of negative constraints $\mathcal{N}'$

Figure 4-3: *Example 4.7*

EXAMPLE 4.7 *Consider the system of negative constraints $\mathcal{N}$ and the p-skyline relation $\succ$ from Example 4.5 (they are shown in Figures 4-3(a) and 4-3(b) correspondingly). The result $\mathcal{N}'$ of minimization of $\mathcal{N}$ w.r.t $\succ$ is shown in Figure 4-3(c). Only the constraint τ'_2 is different from τ_2 because $(A_2,A_3) \in \Gamma_{\succ}$ and $A_2 \in \mathcal{L}_{\tau_2}$, $A_3 \in \mathcal{R}_{\tau_2}$.*

The next proposition summarizes the constraint checking rules over a minimal system of negative constraints.

PROPOSITION 4.3 *Let a full p-skyline relation $\succ \in \mathcal{F}_{\mathcal{H}}$ satisfy a system of negative constraints $\mathcal{N}$, and $\mathcal{N}$ be minimal w.r.t. $\succ$. Let $\succ'$ be such a full p-skyline extension of $\succ$*

that every edge in $\Gamma_{\succ'} - \Gamma_{\succ}$ starts or ends in A. Denote the new *parents and children of A in $\Gamma_{\succ'}$ as P_A and C_A correspondingly. Then $\succ'$ violates $\mathcal{N}$ if and only if there is a constraint $\tau \in \mathcal{N}$ such that*

1. *$\mathcal{R}_\tau = \{A\} \wedge P_A \cap \mathcal{L}_\tau \neq \emptyset$, or*

2. *$A \in \mathcal{L}_\tau \wedge \mathcal{R}_\tau \subseteq C_A$*

PROOF

$\boxed{\Leftarrow}$ Trivial since the two conditions above imply violation of $\mathcal{N}'$ by $\succ$.

$\boxed{\Rightarrow}$ Assume that there is no constraint τ for which the two conditions hold, but some $\tau' \in \mathcal{N}$ is violated, i.e.,

$$Ch_{\Gamma_{\succ}}(\mathcal{L}_{\tau'}) \supseteq \mathcal{R}_{\tau'}.$$

By Theorem 3.4, $\Gamma_{\succ} \subset \Gamma_{\succ'}$. We also know that all the new edges in $\Gamma_{\succ'}$ start or end in A. Since $\Gamma_{\succ}$ satisfies τ' but $\Gamma_{\succ'}$ does not, we get that either $A \in \mathcal{L}_{\tau'}$ or $A \in \mathcal{R}_{\tau'}$. If A is in $\mathcal{R}_{\tau'}$ then the fact that τ' is violated by $\Gamma_{\succ'}$ implies that $\mathcal{R}_{\tau'} = \{A\}$. Moreover, the fact that τ' is minimal w.r.t. $\succ$ implies $P_A \cap \mathcal{L}_{\tau'} \neq \emptyset$. If $A \in \mathcal{L}_{\tau'}$, then the minimality of τ' implies that τ' is violated because of $\mathcal{R}_{\tau'} \subseteq C_A$. □

Proposition 4.3 is illustrated in the next example.

EXAMPLE 4.8 *Take the minimal system of negative constraints $\mathcal{N}'$ w.r.t. $\succ$ from Example 4.7. Let us construct a full p-skyline extensions $\succ'$ of $\succ$ such that every edge in $\Gamma_{\succ'} - \Gamma_{\succ}$ starts or ends in A_4. Consider possible edges going to A_4. We use Proposition 4.3 to check if a new edge violates $\mathcal{N}'$. The edge $(A_1, A_4) \in \Gamma_{\succ'}$ is not allowed because $A_1 \in \mathcal{L}_{\tau'_3}$ and $\{A_4\} = \mathcal{R}_{\tau'_3}$. The edge $(A_3, A_4) \in \Gamma_{\succ'}$ is not allowed because $A_3 \in \mathcal{L}_{\tau'_3}$ and $\{A_4\} = \mathcal{R}_{\tau'_3}$. However, the edge $(A_2, A_4) \in \Gamma_{\succ'}$ is allowed. The p-graph of the resulting $\succ'$ is shown in Figure 4-4. One can analyze the edges going from A_4 in a similar fashion.*

$$\begin{tikzcd}
 & \boxed{A_2} \arrow[dd] \arrow[dr] & \\
\boxed{A_1} & & \boxed{A_4} \\
 & \boxed{A_3} &
\end{tikzcd}$$

Figure 4-4: $\Gamma_{\succ'}$ *from Example 4.8*

4.4.3 p-skyline discovery algorithm

In this section, we show an algorithm for p-skyline relation discovery which exploits the ideas developed in the previous sections. The function `discover` (Algorithm 4.1) is the main function of the algorithm. It takes four arguments: the set of superior examples G, the entire set of tuples O, the set of atomic preferences $\mathcal{H}$, and the set of all relevant attributes $\mathcal{A}$. It returns a normalized syntax tree of an optimal p-skyline relation favoring G in O. Following Proposition 4.1, we require the set G to be a subset of $w_{sky_{\mathcal{H}}}(O)$. First, we construct the set of negative constraints $\mathcal{N}$ for the superior objects G. We start with $sky_{\mathcal{H}}$ as the initial p-skyline relation favoring G in O. After that, we take the set M consisting of a single attribute, and in every iteration, we enlarge it and construct an optimal M-favoring p-skyline relation. As a result, the function returns an optimal p-skyline relation favoring G in O. The construction of an optimal $(M \cup A)$-favoring relation from an optimal M-favoring relation is performed in the `repeat/until` loop (lines 5-8). Here we use the function `push` which constructs a minimal $(M \cup \{A\})$-*favoring extension* of the relation represented by the syntax tree T. It returns *true* if T has been extended to a relation not violating $\mathcal{N}$, and further extensions are feasible. Otherwise, it returns $false$. The syntax tree T passed to `push` has to be normalized. Hence, after extending the relation, we normalize its syntax tree (line 7). To do that, we use the the procedure `normalizeTree` (Algorithm 3.1). The `repeat/until` loop terminates when no further extensions of T not violating $\mathcal{N}$ are possible.

Let us take a closer look at the function `push` (Algorithm 4.2). It takes four arguments: a set M of attributes, a normalized syntax tree T of an M-favoring p-skyline relation, the

Algorithm 4.1 `discover`(G, $\mathcal{O}$, $\mathcal{H}$, $\mathcal{A}$)

```
Require: $G \subseteq w_{sky_{\mathcal{H}}}(\mathcal{O})$
1: $\mathcal{N} = \mathcal{N}(G, \mathcal{O})$
2: $T$ = a normalized syntax tree of $sky_{\mathcal{H}}$
3: $M$ = set containing an arbitrary attribute from $\mathcal{A}$
4: for each attribute $A$ in $\mathcal{A} - M$ do
5:     repeat
6:         $r$ = push($T$, $M$, $A$, $\mathcal{N}$);
7:         normalizeTree(root of $T$);
8:     until $r$ is false
9:     $M = M \cup \{A\}$
10: end for
11: return $T$
```

current attribute A, and a system of negative constraints $\mathcal{N}$ minimal w.r.t. $\succ$ ($\succ$ here is a p-skyline relation represented by the syntax tree T). It returns $true$ if a rewriting rule has been applied to T without violating $\mathcal{N}$, and $false$ if no rewriting rule can be applied to T without violating $\mathcal{N}$. When the function returns, $\mathcal{N}$ is minimal w.r.t. the p-skyline relation represented by the modified syntax tree, and T is normalized.

The goal of `push` is to find an appropriate rewriting rule which adds to the current p-graph edges going from M to A or vice versa. The function has two branches: the parent of the node A in the syntax tree T is a $\&$ -node (i.e., we may apply $Rule_1$ where N_1 is A or $Rule_2$ where N_m is A), or a $\otimes$ -node (i.e., we may apply $Rule_1$ or $Rule_2$ where C_{i+1} is A, or $Rule_3$ where C_i or C_{i+1} is A). In the first branch (line 2-14), we distinguish between applying $Rule_1$ (line 3-8) and $Rule_2$ (line 9-14). It is easy to notice that with the parameters specified above, the rules are exclusive. However, the application patterns are similar. First, we find an appropriate child C_{i+1} of R (lines 4 and 10). It is important for $Var(C_{i+1})$ to be a subset of M because we want to add edges going from M to A or from A to M. Then we check if the corresponding rule application will not violate $\mathcal{N}$. To do that, we use the function `checkConstr` (lines 5 and 11) as per Proposition 4.3. If a rule application does not violate $\mathcal{N}$, we apply the corresponding rule (lines 6 and 12) and minimize $\mathcal{N}$ w.r.t.

the p-skyline which is the result of the rewriting (Proposition 4.2). To do that, we use the function `minimize`.

The second branch of `push` is similar to the first one and different in only the rewriting rules applied. As a result, it is easy to notice that `push` checks every possible rule application not violating $\mathcal{N}$ and adds to the p-graph only the edges going from A to the elements of M or vice versa.

In our implementation of the algorithm, all sets of attributes are represented as bitmaps of fixed size $|\mathcal{A}|$. Similarly, every negative constraint τ is represented as a pair of the bitmaps corresponding to $\mathcal{L}_\tau$ and $\mathcal{R}_\tau$. With every node C_i of the syntax tree, we associate a variable storing $Var(C_i)$. Its value is updated whenever the children list of C_i is changed.

THEOREM 4.4 *The function* `discover` *returns a syntax tree of an optimal p-skyline relation favoring G in O. Its runtime is* $O(|\mathcal{N}| \cdot |\mathcal{A}|^3)$.

PROOF

First, we prove that `discover` always returns a maximal p-skyline relation satisfying $\mathcal{N}$. By construction, a p-skyline relation returned by the function satisfies the system of negative constraints $\mathcal{N}$ constructed in `discover`. Now prove that $\succ$ returned by `discover` is a maximal p-skyline relation satisfying $\mathcal{N}$. A simple case analysis shows that `push` picks every p-skyline relation which is

1. a minimal extension of $\succ$ represented by the parameter T, and

2. whose p-graph has only edges going between the nodes $(M \cup \{A\})$,

until it finds one not violating $\mathcal{N}$ (of course, given the fact that $\succ'$, whose p-graph obtained from $\Gamma_\succ$ by removing edges going to/from A, is optimal M-favoring). Recall that T computed in line 2 of `discover` represents a maximal M-favoring p-skyline relation satisfying $\mathcal{N}$, for a singleton M. Now assume that T at the end of some iteration of the **for**-loop of `discover` represents a not maximal M-favoring p-skyline relation. Take the

Algorithm 4.2 `push`(T, M, A, $\mathcal{N}$)

```
Require: $T$ is normalized
 1: if the parent of $A$ in $T$ is of type &
 2:   $C_i$ := parent of $A$ in $T$; $R$ := parent of $C_i$ in $T$;
 3:   if $A$ is the first child of $C_i$
 4:     for each child $C_{i+1}$ of $R$ s.t. $Var(C_{i+1}) \subseteq M$
 5:       if checkConstr($\mathcal{N}$, $A$, $\emptyset$, $Var(C_{i+1})$)
 6:         apply $Rule_1(T, C_i, C_{i+1})$
 7:         $\mathcal{N} := minimize(\mathcal{N}, Var(A), Var(C_{i+1}))$
 8:         return $true$
 9:   else if $A$ is the last child of $C_i$
10:     for each child $C_{i+1}$ of $R$ s.t. $Var(C_{i+1}) \subseteq M$
11:       if checkConstr($\mathcal{N}$, $A$, $Var(C_{i+1})$, $\emptyset$)
12:         apply $Rule_2(T, C_i, C_{i+1})$
13:         $\mathcal{N} := minimize(\mathcal{N}, Var(C_{i+1}), Var(A))$
14:         return $true$
15: else // the parent of $A$ in $T$ is of type $\otimes$
16:   $R$ := parent of $A$ in $T$;
17:   for each child $C_i$ of $R$ s.t. $Var(C_i) \subseteq M$
18:     if $C_i$ is of type &
19:       $N_1$ := first child of $C_i$, $N_m$ := last child of $C_i$
20:       if checkConstr($\mathcal{N}$, $A$, $Var(N_1)$, $\emptyset$)
21:         apply $Rule_1(T, C_i, A)$
22:         $\mathcal{N}$ :=minimize($\mathcal{N}$, $Var(N_1)$, $Var(A)$)
23:         return $true$
24:       else if checkConstr($\mathcal{N}$, $A$, $\emptyset$, $Var(N_m)$)
25:         apply $Rule_2(T, C_i, A)$
26:         $\mathcal{N}$ := minimize($\mathcal{N}$, $Var(A)$, $Var(N_m)$)
27:         return $true$
28:     else // $C_i$ is a leaf node, since $T$ is normalized
29:       if checkConstr($\mathcal{N}$, $A$, $Var(C_i)$, $\emptyset$)
30:         apply $Rule_3(T, C_i, A)$
31:         $\mathcal{N}$ :=minimize($\mathcal{N}$, $Var(C_i)$, $Var(A)$)
32:         return $true$
33:       else if checkConstr($\mathcal{N}$, $A$, $\emptyset$, $Var(C_i)$
34:         apply $Rule_3(T, A, C_i)$
35:         $\mathcal{N}$ :=minimize($\mathcal{N}$, $Var(A)$, $Var(C_i)$)
36:         return $true$
37: return $false$
```

Algorithm 4.3 `checkConstr`$(\mathcal{N}, A, P_A, C_A)$

```
for each $\tau \in \mathcal{N}$ do
  if $\mathcal{R}_\tau = \{A\} \wedge P_A \cap \mathcal{L}_\tau \neq \emptyset$ or $A \in \mathcal{L}_\tau \wedge \mathcal{R}_\tau \subseteq C_A$ then
    return $false$
  end if
end for
return $true$
```

Algorithm 4.4 `minimize`$(\mathcal{N}, U, D)$

```
1: for each constraint $\tau$ in $\mathcal{N}$ do
2:   if $U \cap \mathcal{L}_\tau \neq \emptyset$ then
3:     $\mathcal{R}_\tau \leftarrow \mathcal{R}_\tau - D$
4:   end if
5: end for
6: return $\mathcal{N}$
```

first such an iteration of the **for**-loop. It implies that there is $\succ^*$ which strictly contains $\succ$ and satisfies $\mathcal{N}$. By Theorem 3.4, $\Gamma_{\succ^*}$ also strictly contains $\Gamma_{\succ}$. Take an edge $(X,Y) \in \Gamma_{\succ^*}$ which is not in $\Gamma_{\succ}$. Let $\succ'$ be the relation computed in the **for**-loop in `discover` when A was equal to X or Y, whatever was the last one. Take the corresponding set M. According to the argument above, $\succ'$ is optimal M-favoring. Since $\succ' \subseteq \succ$, $\Gamma_{\succ'}$ does not contain the edge from X to Y. At the same time, if we take $\Gamma_{\succ^*}$ and leave in it only the edges going to and from the elements of M, it will strictly contain $\Gamma_{\succ'}$ and not violate $\mathcal{N}$. Hence, $\succ'$ is not optimal M-favoring, which is a contradiction. That implies that `discover` returns an optimal $\mathcal{A}$-favoring (or simply favoring) p-skyline relation satisfying $\mathcal{N}$.

Now let us show that the runtime of the algorithm is $O(|\mathcal{N}| \cdot |\mathcal{A}|^3)$. First, let us consider the running time of the subprocedures. The runtime of `minimize` and `checkConstr` is $O(|\mathcal{N}| \cdot |\mathcal{A}|)$. The time needed to modify the syntax tree using any transformation rule is $O(|\mathcal{A}|)$: every rule creates, deletes, and modifies a constant number of nodes of a syntax tree, but updating their Var-variables is done in $O(|\mathcal{A}|)$. Similarly, syntax tree normalization runs in $T_{normalizeTree} = O(|\mathcal{A}|)$ for such modified syntax trees. As a result,

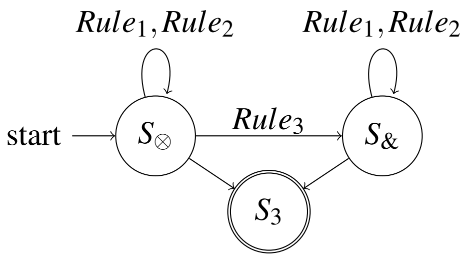


Figure 4-5: *Using* `push` *for computation of an optimal $(M \cup \{A\})$-favoring p-skyline relation*

the time needed to execute the bodies of the loops (lines 5-8, 11-14, 18-36) of `push` is $T_{rule} = O(|\mathcal{N}| \cdot |\mathcal{A}|)$.

Now let T be a syntax tree of an optimal M-favoring p-skyline relation. Now consider the way `push` is used in `discover` to construct an optimal $(M \cup \{A\})$-favoring p-skyline relation. The state diagram of this process is shown in Figure 4-5. It has three states: $S_\otimes$ and $S_\&$ which correspond to T in which A is a child of a $\otimes$- and &-node, respectively; and S_3 which corresponds to the case when no rewriting rule can be applied to T, or any rule application violates $\mathcal{N}$.

The starting state is $S_\otimes$, because in the starting T, A is a child of the topmost $\otimes$-node. After applying the rewriting rules $Rule_1$ and $Rule_2$ in lines 21 and 25 respectively, A becomes a child node of another $\otimes$-node of the modified T. After applying $Rule_3$ (lines 30 and 34), A becomes a child of a &-node in the modified T, and we go to the state $S_\&$. When in $S_\&$, we can only apply $Rule_1$ or $Rule_2$ from lines 6 and 12 respectively. Note that after applying these rules, A is still a child of the same &-node in the modified T. When no rule can be applied to T at any state, we go to the accepting state S_3 and return $false$.

Consider the total number of nodes of T enumerated in the loops (lines 4-8, 10-14, and 17-36) of `push` to construct an optimal $(M \cup \{A\})$-favoring p-skyline relation. Note that when we go from $S_\otimes$ to $S_\otimes$ by applying $Rule_1$ or $Rule_2$, A becomes a descendent of the $\otimes$-node whose child it was originally. Hence, when in $S_\otimes$ we enumerate the nodes C_i to apply $Rule_1$ or $Rule_2$ to, we never pick any C_i which we picked in the previous calls of `push`. In the process of going from $S_\&$ to itself via an application of $Rule_1$ or $Rule_2$, we *may* enumerate the same node C_{i+1} more than once because A does not change its parent &-node as a result of these applications. To avoid checking these rules against the same nodes C_{i+1} more than once, one can keep track of the nodes which have already been picked and tested.

Since the total number of nodes in a syntax tree is $O(|\mathcal{A}|)$, the tests $Var(C_{i+1}) \subseteq M$ (lines 4, 10) and $Var(C_i) \subseteq M$ (line 17) are performed $O(|\mathcal{A}|)$ times and the rules are applied to the tree $O(|\mathcal{A}|)$ times. Each of the containment tests above requires time $O(|\mathcal{A}|)$ given the bitmap representation of sets. Hence, to compute the tree of an optimal $(M \cup \{A\})$-favoring from the tree of an optimal M-favoring p-skyline relation, we need time $O(|\mathcal{N}| \cdot |\mathcal{A}|^2)$. Finally, the running time of `discover` is $O(|\mathcal{N}| \cdot |\mathcal{A}|^3)$. $\square$

An important observation here is that the order in which the attributes are picked and put to M in `discover` may be arbitrary. Moreover, the order of rules in which they are applied in `push` may be also changed. That is, we currently try to apply $Rule_1$ (line 21) first and $Rule_2$ (line 25) after it. However, one can apply them in the opposite order. The same applies to $Rule_3(T,A,C_i)$ and $Rule_3(T,C_i,A)$ (lines 30 and 34, respectively). If any of these parameters of the algorithm is changed, the generated p-skyline relation may be different. However, even if it is different, it will still be an optimal favoring G in O. Note also that due to the symmetry of $\otimes$, the order of child nodes in a $\otimes$ may be different in normalized p-skyline trees of equivalent p-skyline relations. Hence, the order in which the leaf nodes are stored in the normalized syntax tree of $sky_{\mathcal{H}}$ (line 2 of `discover`) also affects the resulting p-skyline relation.

An example of applying Algorithm 4.1 is shown in the next section.

4.4.4 Reducing the number of negative constraints

As we showed in Theorem 4.4, the runtime of the function `discover` linearly depends on the size of the system of negative constraints $\mathcal{N}$. If $\mathcal{N} = \mathcal{N}(G, O)$, then it contains $(|O|-1) \cdot |G|$ constraints. In this section, we propose two methods of reducing the size of $\mathcal{N}$. Both methods are based on applying the skyline operator. However, the first method applies it to the tuple set O, and the second to the set of constraints itself.

Note that each negative constraint is used to show that a tuple should not be preferred to a superior example. We also know that the relation $sky_{\mathcal{H}}$ is the least full p-skyline relation. By definition of the winnow operator, for every $o' \in O - w_{sky_{\mathcal{H}}}(O)$ there is a tuple $o \in w_{sky_{\mathcal{H}}}(O)$ s.t. o is preferred to o' according to $sky_{\mathcal{H}}$. Since $sky_{\mathcal{H}}$ is the least full p-skyline relation, the same o is preferred to o' according to every full p-skyline relation. Thus, in order to guarantee favoring G in O, the system of negative constraints needs to contain only the constraints showing that the tuples in $w_{sky_{\mathcal{H}}}(O)$ are not preferred to the superior examples. The size of such a system of negative constraints is $(|w_{sky_{\mathcal{H}}}(O)|-1) \cdot |G|$.

Another way to reduce the size of a system of negative constraints is based on the following fact. Let us take two negative constraints $\tau, \tau' \in \mathcal{N}$ such that $\mathcal{L}_{\tau'} \subseteq \mathcal{L}_{\tau}$, $\mathcal{R}_{\tau} \subseteq \mathcal{R}_{\tau'}$, and let at least one of these relationships be strict. It is easy to check that τ strictly implies τ'. Thus, the constraint τ' is redundant and may be deleted from $\mathcal{N}$. This idea can also be expressed as follows:

$$\tau \text{ strictly implies } \tau' \text{ if and only if } \mathcal{L}_{\tau'} \subseteq \mathcal{L}_{\tau} \wedge (\mathcal{A} - \mathcal{R}_{\tau'}) \subseteq (\mathcal{A} - \mathcal{R}_{\tau})$$
$$\text{and at least one containment is strict}$$

Let us represent τ as a bitmap representing $(\mathcal{A} - \mathcal{R}_{\tau})$ appended to a bitmap representing $\mathcal{L}_{\tau}$. We assume that a bit is set to 1 iff the corresponding attribute is in the corresponding set. Denote such a representation as $bitmap(\tau)$.

EXAMPLE 4.9 *Let $\mathcal{L}_\tau = \{A_1,A_3,A_5\}$, $\mathcal{R}_\tau = \{A_2\}$, $\mathcal{L}_{\tau'} = \{A_1,A_5\}$, $\mathcal{R}_{\tau'} = \{A_2,A_4\}$. Let $\mathcal{A} = \{A_1,\ldots,A_5\}$. Then $bitmap(\tau) = 10101\ 10111$ and $bitmap(\tau') = 10001\ 10101$.*

Consider $bitmap(\tau)$ as a vector with $2 \cdot |\mathcal{A}|$ dimensions. From the negative constraint implication rule, it follows that τ strictly implies τ' iff $bitmap(\tau)$ and $bitmap(\tau')$ satisfy the *Pareto improvement principle*, i.e., *the value of every dimension of $bitmap(\tau)$ is greater or equal to the corresponding value in $bitmap(\tau)$, and there is at least one dimension whose value in $bitmap(\tau)$ is greater than in $bitmap(\tau')$*. Therefore, the set of all non-redundant constraints in $\mathcal{N}$ corresponds to the *skyline* of the set of bitmap representations of all constraints in $\mathcal{N}$. Moveover, $bitmap(\tau)$ can have only two values in every dimension. Thus, an algorithm for computing skylines over low cardinality domains (e.g. [MPJ07]) can be used to compute the set of non-redundant constraints.

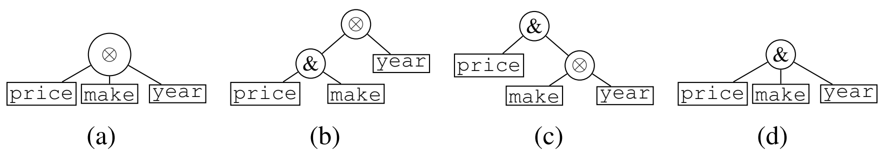


Figure 4-6: *Example 4.10*

EXAMPLE 4.10 *Take O and $\mathcal{H}$ from Example 4.1, and G from Example 4.3. Then the corresponding system of negative constraints $\mathcal{N}(G,O)$ (Example 4.3) is*

$\tau_1 : t_1 \not\succ t_3$	$Ch_{\Gamma_\succ}(\{\texttt{make}\}) \not\supseteq \{\texttt{price}\}$
$\tau_2 : t_2 \not\succ t_3$	$Ch_{\Gamma_\succ}(\{\texttt{make},\texttt{year}\}) \not\supseteq \{\texttt{price}\}$
$\tau_3 : t_4 \not\succ t_3$	$Ch_{\Gamma_\succ}(\{\texttt{make},\texttt{year}\}) \not\supseteq \{\texttt{price}\}$
$\tau_4 : t_5 \not\succ t_3$	$Ch_{\Gamma_\succ}(\{\texttt{make}\}) \not\supseteq \{\texttt{price},\texttt{year}\}$

Let us find an optimal $\succ \in \mathcal{F}_{\mathcal{H}}$ favoring G in O by running `discover`*. Among the constraints above, only $\mathcal{N} = \{\tau_2 : Ch_{\Gamma_\succ}(\{\texttt{make}, \texttt{year}\}) \not\supseteq \{\texttt{price}\}\}$ is nonredundant.*

Moreover, it cannot be further minimized because $\mathcal{R}_\mathcal{I}$ *is a singleton. Consider the attributes in the order:* `make`, `price`, *and* `year`.

We run `discover`. *The tree T (line 2) is shown in Figure 4-6(a). The initial value of M is* $\{$`make`$\}$. *First, we call* `push`$(T, M,$`price`$,\mathcal{N})$. *The parent of* `price` *is the* $\otimes$-*node (Figure 4-6(a)), so we go to line 16 of* `push`, *where R is set to the* $\otimes$-*node (Figure 4-6(a)). After* C_i *is set to the node* `make` *in line 17, we go to line 29 because it is a leaf node. The* `checkConstr` *test in line 29 fails because* $\mathcal{N}$ *prohibits the edge* `make` $\rightarrow$ `price`. *Hence, we go to line 33 where the* `checkConstr` *test succeeds. We apply* $Rule_3(T,$`price`$,C_i)$, `push` *returns true, and the resulting syntax tree T is shown in Figure 4-6(b). Next time we call* `push`$(T, M,$`price`$,\mathcal{N})$ *(line 6 of* `discover`*), we get to line 4 of* `push`. *Since* `year` $\notin M$, *we immediately go to line 37 and return* $false$. *In* `discover` *we set M to* $\{$`make,price`$\}$ *and call* `push`$(T, M,$`year`$,\mathcal{N})$. *There we go to line 16 (R is set to the* $\otimes$-*node in Figure 4-6(b)),* C_i *is set to the &-node (Figure 4-6(b)), we apply* $Rule_1(T, C_i,$`year`$)$ *(the resulting tree T is show in Figure 4-6(c)), and* $true$ *is returned. When push*$(T, M,$`year`$,\mathcal{N})$ *is called next time, we first go to line 16, R is set to the* $\otimes$-*node (Figure 4-6(c)), and* C_i *to the node* `make`. *Then* $Rule_3(T, C_i,$`year`$)$ *is applied (line 30) resulting in the tree T shown in Figure 4-6(d), and* $true$ *is returned.* `push`$(T, M, year, \mathcal{N})$ *is called once again from* `discover`, *but it returns* $false$, *and thus the tree in Figure 4-6(d) is the final one. According to the corresponding p-skyline relation,* t_3 *dominates all other tuples in O.*

The final p-skyline relation constructed in Example 4.10 is a prioritized accumulation of all the atomic preference relations. This is due to the fact that $\mathcal{N}$ contained only one constraint. When more constraints are involved, a discovered p-skyline relation generally also has occurrences of Pareto accumulation.

4.5 Experiments

We have performed extensive experimental study of the proposed framework. The algorithms were implemented in Java. The experiments were ran on Intel Core 2 Duo CPU 2.1 GHz with 2.0GB RAM. The experiments were run on four data sets: one real life and three synthetic.

4.5.1 Experiments with real life data

In this section, we focus on experimenting with the accuracy of the p-skyline relation discovery algorithm and the reduction of skylines achieved by modeling user preferences using p-skyline relations. We used a data set O which stores statistics of NHL players [nhl08] containing 9395 tuples. We used three sets of relevant attributes $\mathcal{A}$ of 12, 9, and 6 attributes, respectively. The sizes of the corresponding skylines were 568, 114, and 33.

Accuracy of p-skyline relation discovery

The aim of the first experiment is to demonstrate that the proposed p-skyline discovery algorithm has a high accuracy. We use the following scenario here. We assume that preferences of a user are modeled as a p-skyline relation, denoted as $\succ_{fav}$. We assume that the user provides the set of relevant attributes $\mathcal{A}$, the corresponding atomic preferences $\mathcal{H}$, and a set G_{fav} of tuples which she likes most in O (i.e., G_{fav} are superior examples and $G_{fav} \subseteq w_{\succ_{fav}}(O)$). We use G_{fav} to compute an optimal p-skyline relation $\succ$ favoring G_{fav} in O. To measure the accuracy of the p-skyline relation discovery algorithm, we compare the set of the best tuples $w_{\succ}(O)$ in O according to the computed p-skyline relation $\succ$ with the set of the best tuples $w_{\succ_{fav}}(O)$ in O according to the preference relation $\succ_{fav}$ (which, as we assumed, correctly describes the user preferences).

To model user preferences, we randomly generated 100 p-skyline relations $\succ_{fav}$. For each $w_{\succ_{fav}}(O)$, we randomly picked 5 tuples from it, and used them as superior examples G_{fav} to discover three different optimal p-skyline relations $\succ$ favoring G_{fav} in O. Out of those three relations, we picked the one resulting in $w_{\succ}(O)$ of the smallest size. Then we added 5 more tuples from $w_{\succ_{fav}}(O)$ to G_{fav} and repeated the same procedure. We kept adding tuples to G_{fav} from $w_{\succ_{fav}}(O)$ until G_{fav} reached $w_{\succ_{fav}}(O)$.

To measure the accuracy of the p-skyline discovery algorithm, we computed the following three values: (i) *precision* of the p-skyline discovery method

$$precision = \frac{|w_{\succ}(O) \cap w_{\succ_{fav}}(O)|}{|w_{\succ}(O)|},$$

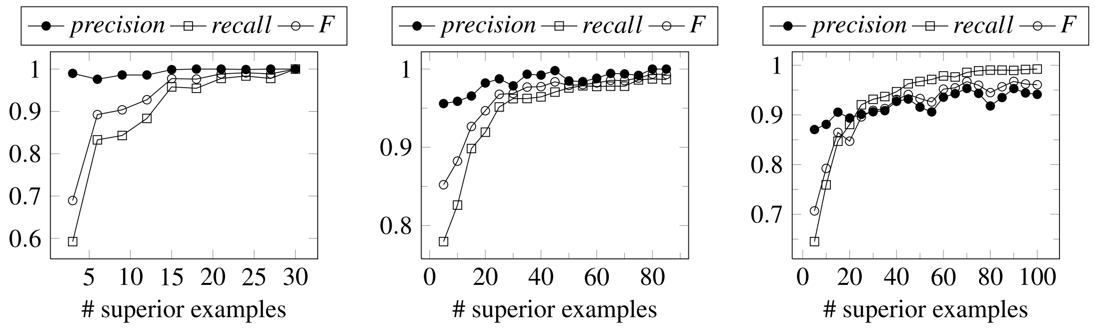
(ii) *recall* of the p-skyline discovery method

$$recall = \frac{|w_{\succ}(O) \cap w_{\succ_{fav}}(O)|}{|w_{\succ_{fav}}(O)|},$$

and (iii) *F-measure* which combines *precision* and *recall*

$$F = 2 \cdot \frac{precision \cdot recall}{precision + recall}$$

We plotted the average values of these measures in Figure 4-7. As can be observed, *precision* of the p-skyline relation discovery algorithm is high in all experiments. In particular, it is greater than 0.9 in most cases regardless of the number of superior examples used to discover a p-skyline relation and the number of relevant attributes. At the same time, *recall* of the p-skyline relation discovery method starts from a low value when the number of superior examples is low. That is justified by the fact that the proposed discovery algorithm computes an *optimal* relation favoring G_{fav} in O. Thus, when G_{fav} contains few tuples, it is not sufficient to capture the corresponding preference relation $\succ_{fav}$, and thus the ratio of false negatives is rather high. However, when we increase the number of superior examples, *recall* consistently grows.



(a) $|\mathcal{A}| = 6$ (b) $|\mathcal{A}| = 9$ (c) $|\mathcal{A}| = 12$

Figure 4-7: *Accuracy of p-skyline discovery*

In Figure 4-8, we plot the value of F-measure with respect to the share of the skyline used as superior examples to discover the relation $\succ$. As one can observe, the value of F starts from a comparatively low value of 0.7 but quickly reaches 0.9 via a small increase of the size of G_{fav}. Another important observation is that the value of F is generally inversely dependent on the number of relevant attributes (given the same ratio of superior examples used). This is justified by the following fact. In order to compute a p-skyline relation favoring G_{fav} in O, the algorithm uses a set of negative constraints $\mathcal{N}$. Intuitively, the computed p-skyline relation $\succ$ will match the original relation $\succ_{fav}$ better if the set $\mathcal{N}$ "describes" $\succ_{fav}$ sufficiently well. The number of constraints in $\mathcal{N}$ depends not only on the number of superior examples but also on the total number of tuples in the corresponding skyline. Since skyline sizes are generally smaller for smaller sets of $\mathcal{A}$, more superior objects are needed for smaller $\mathcal{A}$ to "describe" $\succ_{fav}$.

Reduction of size of winnow query result

In Section 3.8, we described a well known deficiency of the skyline framework – skylines are generally of large size for large sets of relevant attributes. The goal of the experiments in this section is twofold. First, we demonstrate that using p-skyline relations to model user preferences results in *smaller winnow query results* in comparison to skylines. Second,

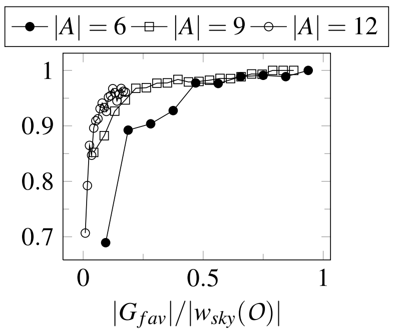


Figure 4-8: *F-measure*

we show that the reduction of query result size is significant if the *actual* user preference relation (i.e., the user preference relation we model) is a p-skyline relation. In particular, we show it is generally hard to find a p-skyline relation accurately describing *an arbitrary* subset of skyline.

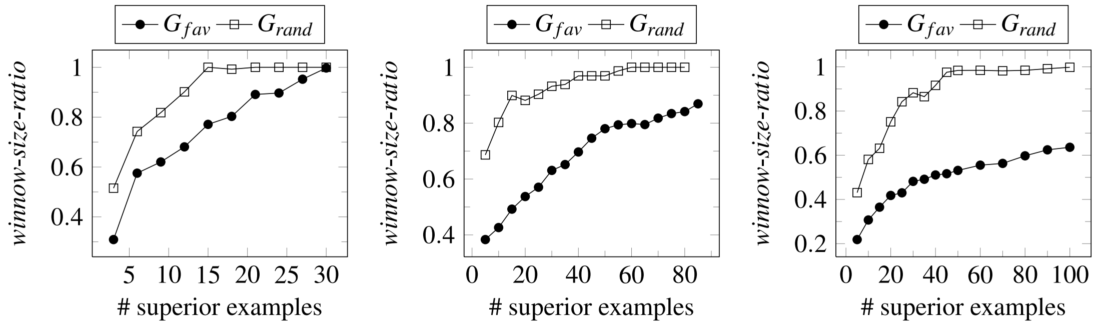
In this experiment, sets of superior examples are generated using two methods. First, they are drawn randomly from the set of the best objects $w_{\succ_{fav}}(O)$ according to a real p-skyline relation $\succ_{fav}$. Such sets are generated as in the previous experiment and denoted as G_{fav}. Second, they are drawn randomly from the corresponding skyline $w_{sky}(O)$. Such sets are denoted as G_{rand}. Notice that G_{rand} may not be favored by any p-skyline relation (besides $sky_{\mathcal{H}}$, or course). We use these sets to discover p-skyline relations $\succ$ favoring them. In Figure 4-9, we plot

$$winnow\text{-}size\text{-}ratio = \frac{|w_{\succ}(O)|}{|w_{sky_{\mathcal{H}}}(O)|},$$

which shows the difference in the sizes of results of p-skyline and skyline queries.

Consider the graphs for G_{fav}. As the figure suggests, using p-skyline relations to model user preferences allows for significant reduction in winnow query result size in comparison to skyline. It can be observed that using larger sets of relevant attributes generally results in smaller *winnow-size-ratio* values. Moreover, for larger relevant attribute sets, *winnow-size-ratio* grows slowly. That is due to larger skyline sizes for such sets.

Another important observation is that *winnow-size-ratio* is always smaller for superior examples which correspond to p-skyline relations (G_{fav}) in comparison to superior examples drawn randomly (G_{rand}) from skyline. The fact that superior examples correspond to a real p-skyline relation implies that they share some similarity expressed as the corresponding attribute importance relationships. For a set of random skyline tuples G_{rand}, such similarity exists when it contains only a few tuples. Increasing the size of such a set decreases the similarity of the tuples, which results in a quick growth of *winnow-size-ratio*.



(a) $|\mathcal{A}| = 6$ (b) $|\mathcal{A}| = 9$ (c) $|\mathcal{A}| = 12$

Figure 4-9: *p-skyline size reduction*

4.5.2 Experiments with synthetic data

In this section, we present experiments with synthetic data. The main goals of the experiments is to demonstrate that the proposed p-skyline relation discovery approach has a high scalability and is well optimizable. We used three synthetic data sets O here: correlated, anticorrelated, and uniform. Each of them contained 50000 tuples. We used three different sets $\mathcal{A}$ of 10, 15, and 20 relevant attributes. For each of them, we picked different sets of superior examples G. We constructed such sets of *similar* tuples, where similarity was measured in terms of Euclidean distance. Given a set G, we used `discover` to compute optimal p-skyline relations $\succ$ favoring G.

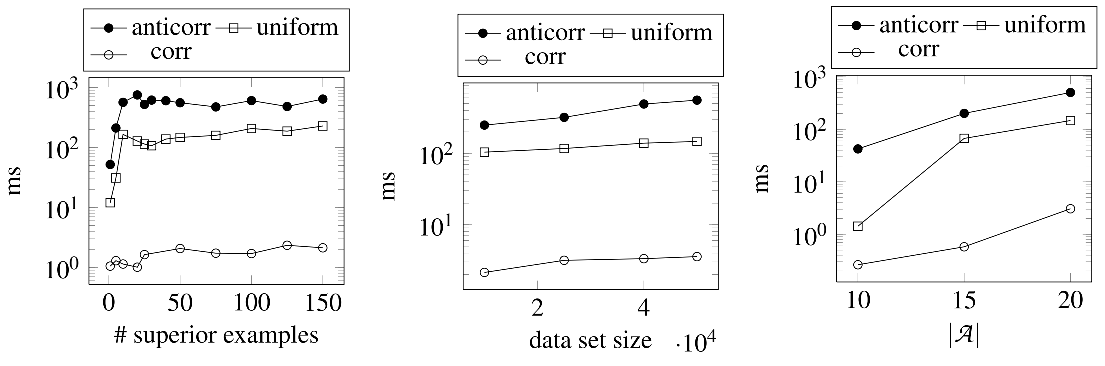
Scalability of p-skyline discovery

In this section, we show that the algorithm we proposed for p-skyline relation discovery is scalable with respect to various parameters of the algorithm. In Figure 4-10, we plot dependency of the average running time of `discover` on the number of superior examples $|G|$ used to discover a p-skyline relation (Figure 4-10(a), $|O| = 50000$, $|\mathcal{A}| = 20$), the size of O (Figure 4-10(b), $|G| = 50$, $|\mathcal{A}| = 20$), and the number $|\mathcal{A}|$ of relevant attributes (Figure 4-10(c), $|O| = 50000$, $|G| = 50$). The measured time does not include the time to construct the system of negative constraints and find the non-redundant constraints in it. According to our experiments, the preprocessing time predominantly depends on the performance of the skyline computation algorithm.

According to Figure 4-10(a), the algorithm running time increases until the size of G reaches 30. After that, it does not vary much. This is due to the fact that the algorithm performance depends on the number of used negative constraints. We use only *non-redundant* constraints for discovery. As we show further, the dependence of the size of a system of non-redundant constraints on the number of superior examples has a pattern similar to Figure 4-10(a).

The growth of running time with the increase in the data set size (Figure 4-10(a)) is justified by the fact that the number of negative constraints depends on skyline size (Section 4.4.4). For the data sets used in the experiment, skyline sizes grow with the sizes of the corresponding data sets. Similarly, the running time of the algorithm grows with the number of relevant attributes (Figure 4-10(c)) due to the increase in the corresponding skyline size.

We conclude that the the running time of the algorithm for p-skyline discovery we have proposed in this chapter has a low running time and scales well with respect to the number of superior examples, the size of the data set, and the number of relevant attributes used in discovery.



(a) against # superior examples (b) against data set size (c) against $|\mathcal{A}|$

Figure 4-10: *Performance of p-skyline discovery*

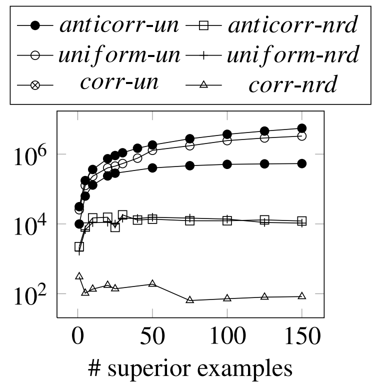
Reduction in the number of negative constraints

In this section, we demonstrate that the proposed approach for p-skyline relation discovery is well optimizable. We showed that the running time of `discover` linearly depends on the number of negative constraints in the system $\mathcal{N}$. Here we show that the techniques proposed in Section 4.4.4 allow for significant reduction in the size of $\mathcal{N}$.

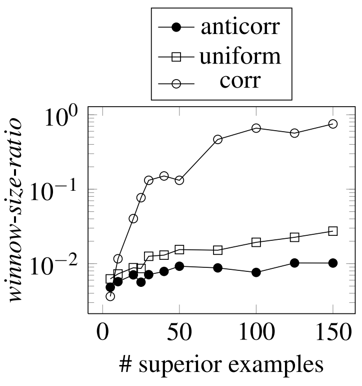
In Figure 4-11(a), we show how the number of negative constraints depends on the number of superior examples used to construct them. For every data set, we plot two values: the number of *unique* negative constraints in $\mathcal{N}(G, w_{sky_{\mathcal{H}}}(O))$ (*anticorr-un*, *uniform-un*, and *corr-un*, resp.) and the number of *unique non-redundant* constraints in the corresponding system (*anticorr-nrd*, *uniform-nrd*, and *corr-nrd*, resp.). We note that the reduction in the constraint number achieved using the methods we proposed in Section 4.4.4 is significant. In particular, for the anticorrelated data set and G of size 150, the total number of constraints in $\mathcal{N}(G, O)$ was approximately $7.5 \cdot 10^6$. Among them, about $5.5 \cdot 10^6$ were unique in $\mathcal{N}(G, w_{sky_{\mathcal{H}}}(O))$. However, less than 1% of them (about $12 \cdot 10^3$) were non-redundant.

Reduction of size of winnow query result

In Section 4.5.1, we showed how that the size of p-skyline preference query result depends on the number of relevant attributes and the size of skyline. In this section, we show that another parameter which affects winnow query size is the data distribution. In Figure 4-11(b), we show how the size of the p-skyline query result varies with the number of superior examples used to discover a p-skyline relation $\succ$. We compare it with the size of the corresponding skyline and plot the value of *winnow-size-ratio* defined in the previous section. Here we used the anticorrelated, the uniform, and the correlated data sets of 50000 tuples each. The numbers of relevant attributes were 20. The sizes of the corresponding skylines were: 41716 (anticorrelated), 37019 (uniform), and 33888 (correlated). For the anticorrelated and the uniform data sets, values of *winnow-size-ratio* quickly reach a certain bound and then grow slowly with the number of superior examples. This bound is approximately 1% of the skyline size (i.e., about 350 tuples) for both data sets. At the same time, the growth of *winnow-size-ratio* for the correlated data set is faster. Note that the values of *winnow-size-ratio* are generally lower for synthetic data sets in comparison to the real life data set in the previous section. This is due to the larger set of relevant attributes and larger skyline sizes in the current experiment.



(a) Constraint number reduction



(b) p-skyline size reduction

Figure 4-11: *Synthetic data experiments*

We conclude that the experiments we carried out show that incorporating attribute importance into skyline relations allows for significant reduction in query result size. The algorithm for p-skyline relation discovery has good scalability in terms of data set size and number of relevant attributes. The algorithm has a high accuracy even for small sets of superior examples used to discover p-skyline relations.

4.6 Related work

An approach of mining preferences aggregated using the accumulation operators was proposed by Holland et al in [HEK03]. Web server logs were used there to mine preference relations. The mining approach is based on statistical properties of log data – more preferable tuples appear more frequently. The process of mining user preferences is split into two parts: mining atomic preferences and mining the accumulation operators connecting the atomic preferences. Atomic preferences are mined in the form of the set of predefined preference constructors such as LOWEST, HIGHEST etc. Categorical domain preferences are mined in the form of simple constructors like POS, NEG as well as arbitrary strict partial order relations. In order to mine preference relations aggregating atomic preferences using Pareto and prioritized accumulation operators, [HEK03] proposed a heuristic approach based on associative rule mining. The cases when more than one different combination of accumulation operators may be mined on the same data were not addressed. Moreover, no criteria of optimality of mined preference relations were defined.

A framework of preference discovery which is a complementary to the approach we have proposed here was presented by Jiang et al in [JPL$^+$08]. In that work, preferences are modeled as skyline relations. Given a set of relevant attributes and a set of atomic preferences over some of them, the problem addressed there is computing atomic preferences over the remaining attributes. The mining process is based on user feedback in terms of a set of superior and a set of inferior examples. The focus of that work is to mine minimal (in terms of relation size) attribute preference relations. It was shown there that the problem of existence of such relations is NP-complete in general, and the computation problem is in general NP-hard. Two heuristic algorithms for computing such preferences were provided. The algorithms are greedy and not sound, i.e., for some inputs, the computed preferences may be not minimal. That approach and the approach we presented here are different in the following sense. First, [JPL$^+$08] deals with skyline relations, and thus all attribute preferences are considered to be equally important. In contrast, the focus of our work is to discover differences in attribute importance. Second, [JPL$^+$08] focuses on mining minimal attribute preferences. In contrast, we are interested in computing maximal preference relations since they guarantees a better fit to a provided set of superior examples. At the same time, our work and [JPL$^+$08] complement each other. Namely, when attribute preferences are not provided explicitly be user, the approach [JPL$^+$08] may be used to discover them.

Another approach of preference relation discovery in the skyline framework was proposed by Lee et al in [LwYwH$^+$08]. That work is motivated by the problem of large skyline sizes. It proposes to reduce skyline sizes by revising skyline preference relations by supplying additional tuple relationships: preference and equivalence. The relationships are obtained from user's answers to simple questions: 1) is an object preferred to another object or vice versa, and 2) are two objects equivalent. [LwYwH$^+$08] also developed an algorithm that given a number k computes a minimal set of questions such that positive answers to them result in the winnow query size at most k.

In quantitative preference frameworks [Fis70], preferences are represented as *utility functions*: a tuple t is considered to be preferred to another tuple t' if $f(t) > f(t')$ for a utility function f. Attribute priorities are often represented here as *weight coefficients* in polynomial utility functions. A number of methods have been proposed to elicit such utility functions [CKP00, Bou02, GW05].

Representing preferences as utility functions was shown to be a powerful tool to reason as well as query databases with preferences [DFP03, BG96, FLN01, DGKT06]. Hence, some work has been performed to eliciting utility functions for preferences represented in other models. In [MD02], preferences are considered over finite domains and under the *ceteris paribus* semantics. That work introduces a two-step algorithm for computing a utility function representing a set of preference statements. It has the following steps: (i) convert the statements to a special graph, and (i) use an appropriate class of a utility function to represent the graph. [MD02] introduced four classes of such utility functions with different properties.

Another model of preference elicitation in the form of utility functions was introduced by Domshlak et al in [DJ07]. That work shows a framework of computing a utility function consistent with a set of comparative (e.g., "A is better than B" or "A is as good as B") statements about preferences. [DJ07] also mentions that the model can also handle classificatory (e.g., "A is good", "B is bad") statements. That approach does not rely on any structure of preference relations. In contrast to the majority of works in this area, preferences over attributes here are not required to be independent, and the interpretations of preference statements is not fixed and is one of the parameters of the elicitation algorithm. The key idea of the framework is to map preference statements about tuples into similar statements about higher dimensional tuples. The dimensionality extension is used to capture the dependence of attribute preferences. After that, the corresponding statements are converted to linear inequality constraints, and machine learning techniques are used to efficiently compute a function satisfying the constraints.

In [VH99], Ha et al proposed an approach of composing binary preference relations and utility functions of the class of *multi-linear* utility functions. They convert preference instances of this class into inequality constraints, use Convex Cone Analysis to compute a finite representation of the the resulting preference relations, and propose an algorithm for reasoning with such preferences.

A quantitative framework of eliciting binary preference relations was presented by Haddawy et al in [HRGM03]. Preferences here are elicited as a knowledge based artificial neural network (KBANN). Prior knowledge of user preferences in terms of pairs of comparative tuple statements is encoded into a network as link weights. A network has $2n$ inputs (where n is the number of attributes in a tuple) on which it takes the values of attributes of two tuples to compare. It outputs a binary value: 1 if the first tuple is preferred to the second, and 0 otherwise. However, properties of the binary relations induced by such networks are not studied in this work. In particular, it is not clear if such relations are SPO. To train a network, a wide range of types of data may be used: numeric ratings, pairwise comparisons etc. [HRGM03] presents an experimental evaluation of the framework and shows its effectiveness.

A number of incremental preference elicitation frameworks based on *example critique* were proposed [LHL97, SL01, VFP06]. This class of frameworks generally uses the following scenario. An Agent starts with little or no information about preferences of the User. In every iteration, the Agent shows to the user a carefully selected set of optimal tuples based on the User preferences learned so far. After that, the User selects the tuples in the set which she does not like and describes what exactly she does not like in them. Given that knowledge, the Agent recomputes the model representing the preferences of the User and goes to the next iteration. This process terminates when the User is completely satisfied with the set of tuples. Preferences in such frameworks are generally represented as utility functions. An important aspect of such systems is the type of user critiques collected from the User and the user interface used to gather such information.

The discussion of the framework of p-skyline relation discovery based on superior/inferior examples which we presented in this chapter is an extended version of our paper [MC09].

Chapter 5

Hierarchical CP-nets

In this chapter, we propose the framework of hierarchical CP-nets (HCP-nets). It is a variant of CP-nets which addresses some problems of the original framework. First, conditionality of preferences over attributes in HCP-nets induces difference in the relative importance of them, which is not always the case in CP-nets. Second, orders induced by HCP-nets are succinctly representable as preference formulas, which is an open question for CP-net. Third, HCP-nets can deal with finite as well as infinite domains, whilst CP-nets are defined for finite domains.

5.1 CP-network framework

The CP-network framework was initially proposed in [BBHP99]. In this approach, preferences can be represented as conditional preference graphs. The nodes of such graphs are attributes, i.e. members of $\mathcal{A}$, and the edges are used to show that a preference over an attribute is *conditioned* on the values of its parents in the graph. The framework can be defined as follows.

The domains of all attributes $\mathcal{A}$ in this framework are considered to be finite. The key notion is the *conditional preference table*. Given an attribute $A \in \mathcal{A}$, the conditional preference table (or CPT) of A is

$$CPT_A = (\Phi_A, U_A),$$

where $U_A \subseteq \mathcal{A} - \{A\}$ and $\Phi_A = \{q_1, \ldots, q_k\}$. Semantically, U_A is a set of the attributes that preferences over A are conditioned on, and Φ_A is a set of conditional preference statements over A. Every q in Φ_A is defined as $u_q : R_q$ where $u_q \in \mathcal{U}_{U_A}$, and R_q is a *total order* over $\mathcal{D}_A$. Semantically, u_q is a *conditional* part of q, while R_q is a *preferential* part of q. We also assume that

- for every $u \in \mathcal{U}_{U_A}$, there is $q \in \Phi_A$ such that $u_q = u$;
- for every pair of different $q_1, q_2 \in \Phi_A$, we have $u_{q_1} \neq u_{q_2}$.

The *order induced by* $q \in \Phi_A$ is defined as

$$q^* = \{(o, o') \mid o.U_A = o'.U_A = u_q \wedge (o.A, o'.A) \in R_q \wedge o.Y_A = o'.Y_A\},$$

where $Y_A = \mathcal{A} - (\{A\} \cup W_A)$. In other words, a tuple o is preferred to a tuple o' according to q^* if $o.A$ is preferred to $o'.A$ according to R_q, the values of U_A in both tuples are equal to u_φ and the values of the remaining attributes are pairwise equal. Note that for every pair of objects $(o, o') \in q^*$, o and o' are different only in A. The principle of comparing objects which are different in a single attribute is commonly referred as *ceteris paribus*.

The *order induced by a conditional preference table* CPT_A is defined as

$$CPT_A^* = \bigcup_{q \in \Phi_A} q^*$$

A *CP-net* is defined as $N = \{CPT_A \mid A \in \mathcal{A}\}$. The *order* $\succ_N$ *induced by a CP-net* N is

$$\begin{tikzcd}
\boxed{make} \arrow[d] \\
\boxed{color}
\end{tikzcd}$$

make
GM $\succ$ *Ford*

make	*color*
GM	*red* $\succ$ *blue*
Ford	*blue* $\succ$ *red*

(a) Υ_{N_1}

$$\begin{tikzcd}
(GM, red) \arrow[d] \arrow[dd, bend left] \arrow[ddd, bend right] \\
(GM, blue) \arrow[d] \arrow[dd, bend right] \\
(Ford, blue) \arrow[d] \\
(Ford, red)
\end{tikzcd}$$

(b) Dominance according to N_1

Figure 5-1: *CP-net from Example 5.1*

defined as shown below. A CP-net is called *consistent* if it induces an SPO.

$$\succ_N = TC(\bigcup_{A \in \mathcal{A}} CPT_A^*).$$

The representation of a CP-net N as a *conditional preference graph* Υ_N is defined as follows. Υ_N is a directed graph whose nodes are attributes, i.e., members of $\mathcal{A}$. Every node A is annotated with the corresponding conditional preference table and has incoming edges from every member of U_A. Although cyclicity of such a graph does not necessary imply inconsistency of the corresponding CP-net, we consider CP-nets with acyclic graphs only.

EXAMPLE 5.1 *Consider a CP-net N_1 representing preferences over cars with two attributes:* make *and* color. *Mary prefers* GM *to* Ford. *Given two* GM *cars, she prefers the* red *car to the* blue *car. Given two* Ford *cars, she prefers the* blue *one to the* red *one. The corresponding CP-net N_1 can be represented by the conditional preference graph Υ_{N_1} shown in Figure 5-1(a). In Figure 5-1(b), we show the order induced by N_1.*

5.1.1 Attribute importance in CP-nets

One of the main concepts of the CP-net framework is the conditionality of preferences over an attribute on the values of a set of attributes. It was pointed out in [BBD$^+$04] that a possible reason for a user to have conditional preferences is a *difference in relative importance of attributes*. Take a preference statement from Example 5.1: Mary prefers *GM* to *Ford*, and given two *GM* with different colors, the *red* one is preferred to the *blue* one. One way of interpreting this sentence is that given a set of cars, their make should be considered first; and only when no benefits in make can be achieved, their colors have to be examined. With such an interpretation, an edge from *make* to *color* in a conditional preference graph also implies that the attribute *make* is more important than the attribute $color$. For instance, according to N_1 (Example 5.1), every *GM* is preferred to every *Ford* regardless of $color$. However, as also noted in [BBD$^+$04], such attribute importance relationships do not always hold. Example 5.2 illustrates such a case.

EXAMPLE 5.2 *Consider the CP-net N_1 in Example 5.1. We extend it by expressing additional preferences over car* size*: given two red cars, Mary prefers the* smaller *one; while between two blue cars, she prefers the* larger *one. The resulting CP-net N_2 is shown in Figure 5-2(a), and the order induced by N_2 is in Figure 5-2(b) (transitive preference edges are skipped for clarity). Note that $(GM, red, small) \not\succ_{N_2} (Ford, blue, large)$.*

As can be observed from Example 5.2, not every *GM* is preferred to every *Ford* despite the fact that *make* is the topmost node in the conditional preference graph. The reason is that due to the *ceteris paribus* semantics of CP-nets, one tuple o is preferred to another o' if and only if there is a chain of tuples $\{o = o_1, o_2, \ldots, o_{k-1}, o_k = o'\}$ (called *flipping sequence* [BBHP99]) such that for every $i \in [1, k-1]$, o_i is preferred to o_{i+1} according to the order CPT_A^* induced by the conditional preference table CPT_A of some attribute A. Moreover, each o_i and o_{i+1} must be different in the value of A only. As Figure 5-2(b) shows, no such a sequence exists for the pair of tuples shown in Example 5.2.

$$\begin{tikzcd}
make \arrow[d] \\
color \arrow[d] \\
size
\end{tikzcd}$$

make
GM $\succ$ *Ford*

make	*color*
GM	*red* $\succ$ *blue*
Ford	*blue* $\succ$ *red*

color	*size*
red	*small* $\succ$ *large*
blue	*large* $\succ$ *small*

(a) Υ_{N_2}

$$\begin{tikzcd}
(GM, red, small) \arrow[r] \arrow[d] \arrow[dd, bend right] & (Ford, red, small) \arrow[d] \\
(GM, red, large) \arrow[r] \arrow[dd, bend right] & (Ford, red, large) \\
(GM, blue, small) \arrow[r] & (Ford, blue, small) \arrow[uu, bend right] \\
(GM, blue, large) \arrow[r] \arrow[u] & (Ford, blue, large) \arrow[u] \arrow[uu, bend right]
\end{tikzcd}$$

(b) Dominance according to N_2 (transitive edges omitted)

Figure 5-2: *CP-net from Example 5.2*

5.1.2 Querying databases with CP-nets

We note that the problem of querying databases with preferences is generally not considered in the CP-net framework. However, in order to use a CP-net in a winnow query, one needs to construct a formula representing the order induced by the net. Some attempts have been made by database researchers to develop algorithms for computing preference formulas for CP-nets [CS05, EK06]. We discuss them in Section 5.6. However, such algorithms are incomplete – they terminate for a limited class of preference formulas. Moreover, one typically obtains exponential bounds on the resulting formula size.

An important distinction between CP-nets and the binary relation preference framework is that the former one is defined for *finite* domains while the latter one for *infinite* domains. Hence, to use CP-nets in the binary relation preference framework, it is necessary to extend CP-nets to infinite domains.

5.2 HCP-nets

In the HCP-net framework, we address the issues listed above: conditional preference graphs capture variable importance of attributes, and infinite domains can be handled. Moreover, we further show an approach of computing polynomial preference formulas representing orders induced by HCP-nets.

HCP-nets are defined similarly to CP-nets: the main concepts of CP-nets such as conditional preference table and conditional preference graph are present in the HCP-net framework. However, the semantics of these concepts in the two frameworks are somewhat different.

Given a set of attributes $\mathcal{A}$, we assume that their domains may be finite as well as infinite. Let a *conditional preference table* CPT_A associated with an attribute $A \in \mathcal{A}$ be defined as a triple

$$CPT_A = (\Phi_A, W_A, U_A)$$

in which $W_A \subseteq \mathcal{A} - \{A\}$, $U_A \subseteq \mathcal{A} - \{A\}$ such that $U_A \cap W_A = \emptyset$ and $\Phi_A = \{q_1, ..., q_k\}$. We note that conditional preference tables in CP-nets (Section 5.1) and HCP-nets are defined similarly. The only difference between them is that CPT_A in HCP-net has an additional component W_A. Semantically, it is the set of attributes which are *less important* than A.

Let $q \in \Phi_A$ be defined as $u_q : R_q$ where u_q is a relation such that $u_q \subseteq \mathcal{D}_{U_A}$, R_q is a strict partial order over $\mathcal{D}_A$, and for all pairs of different $q_1, q_2 \in \Phi_A$, we have $u_{q_1} \cap u_{q_2} = \emptyset$. For $q \in \Phi_A$, the *relation induced by* q is defined as

$$q^* = \{(o, o') : o.U_A = o'.U_A \in u_q \wedge o.Y_A = o'.Y_A \wedge (o.A, o'.A) \in R_q\}$$

where $Y_A = \mathcal{A} - (\{A\} \cup W_A)$. As in the case of CP-nets, q can be viewed as a *conditional preference*: a tuple is preferred to another according to the corresponding preference R_q over A if the values of U_A are in u_q and the values of Y_A are pairwise equal. A difference between the CP-nets is that here the preferences over the attribute A are *independent* of the attributes W_A. In other words, instead of the CP-network principle *everything else being equal* we use the *everything else being equal except for the attributes* W_A principle, which was introduced in [Wil04]. Another important distinction is that q in HCP-nets is a set (rather than a distinct assignment in CP-nets), which makes it possible to handle *infinite domains* in HCP-nets.

The *order induced by a conditional preference table* CPT_A is defined as

$$CPT_A^* = \bigcup_{q \in \Phi_A} q^*$$

PROPOSITION 5.1 *Given a conditional preference table* CPT_A*,*

1. *for all* $q \in \Phi_A$*,* q^* *is an SPO;*

2. CPT_A^* *is an SPO;*

PROOF

1. The SPO of q^* follows from the SPO of R_q.
2. We showed above that for every $q \in \Phi_A$, q^* is an SPO. Given any two different $q_1, q_2 \in \Phi_A$, $range_{q_1^*} \cap \ range_{q_2^*} = \emptyset$ due to $u_{q_1} \cap \ u_{q_2} = \emptyset$. Hence, the union CPT_A^* is an SPO. $\square$

We summarize the notation introduced so far. Take a conditional preference table $CPT_A = (\Phi_A, W_A, U_A)$ and two tuples o and o'. Then W_A is the set of attributes whose values in o and o' do not matter for o to be preferred to o' according to CPT_A^*. On the other hand, the set Y_A contains the attributes whose values need to be pairwise equal in o and o' for such a dominance to hold.

We define now hierarchical CP-networks. Let $\Upsilon(A) = \{(B,A) \mid B \in U_A\}$. $\Upsilon(A)$ can be viewed as a directed graph with incoming edges going from the attributes $B \in U_A$ to a single attribute A. These edges correspond to the conditionality of the preference of attribute A on attributes $B \in U_A$. Then we define the *conditional preference graph of* N as $\Upsilon_N = \bigcup_{A \in \mathcal{A}} \Upsilon(A)$.

DEFINITION 5.1 *A set of conditional preference tables* $N = \{CPT_A : A \in \mathcal{A}\}$ *is called a* hierarchical CP-network *or an* HCP-net *if for every* $A \in \mathcal{A}$,

$$W_A = \bigcup_{B \in Ch_{\Upsilon_N}(A)} (\{B\} \cup W_B).$$

Definition 5.1 also implies that a conditional preference graph of a hierarchical network is *acyclic*, otherwise W_A of some attribute A involved in a cycle would contain A, leading to a contradiction.

Now we define the order *induced by an HCP-net*. It is defined similarly as in CP-nets.

DEFINITION 5.2 *The order* $\succ_N$ *induced by an HCP-net is*

$$\succ_N = TC\left(\bigcup_{A \in \mathcal{A}} CPT_A^*\right)$$

By Definition 5.2, a tuple o is preferred to another tuple o' if there is a sequence of tuples started by o, ended by o', and each tuple in the sequences is preferred to the next one according to some conditional preference table. This principle is captured in the notion of *derivation sequence* we define here. It is defined similarly to the corresponding notion in Chapter 3. Let a tuple o and o' be two tuples, and N be an HCP-net. A *derivation sequence* of $o \succ_N o'$ is a pair $(\Sigma_{o,o'}, \Psi_{o,o'})$, where $\Sigma_{o,o'} = (o = o_1, o_2, \ldots, o_k, o_{k+1} = o')$ and $\Psi_{o,o'} = (A_{i_1}, \ldots, A_{i_k})$ such that

$$CPT^*_{A_{i_1}}(o_1, o_2), \ldots, CPT^*_{A_{i_k}}(o_k, o_{k+1})$$

Another important implication of Definition 5.1 is the relationship between conditional preference graph and sets W_A and Y_A shown in the next corollary.

COROLLARY 5.1 *For any HCP-net N and any* $CPT_A = (\Phi_A, W_A, U_A)$ *of N,*

- $W_A = Desc_{\Upsilon_N}(A)$*;*
- $Y_A = Anc_{\Upsilon_N}(A) \cup Sibl_{\Upsilon_N}(A)$.

Corollary 5.1 implies that in order for a tuple o to be preferred to o' according to CPT^*_A, their values of ancestors and siblings of A in the conditional preference graph have to be pairwise equal. Moreover, the values of the descendants of A in o and o' do not matter for such a dominance to hold. Corollary 5.1 is also useful to show that the orders induced by HCP-nets are *preference relations*, i.e., SPO.

THEOREM 5.1 *The order induced by an HCP-net is an SPO*

PROOF

Take an HCP-net N. By definition, $\succ_N$ is transitive. Now we prove its irreflexivity. For the sake of contradiction, assume $o \succ_N o$ for some tuple o. Take a derivation sequence $(\Sigma_{o,o}, \Psi_{o,o})$ for $o \succ_N o$, where $\Sigma_{o,o} = \{o = o_1, \ldots, o_{k+1} = o\}$ and $\Psi_{o,o} = \{A_{i_1}, \ldots, A_{i_k}\}$. Let a node A be a topmost node of $\Psi_{o,o}$ in Υ_N (the existence of A is guaranteed by the acyclicity of Υ_N). Note that for any $j \in [1,k]$, o_{i_j} and $o_{i_{j+1}}$ may be different in only $Desc\text{-}self_{\Upsilon_N}(A_{i_j})$. Hence the values of $Pa_{\Upsilon_N}(A)$ in all $\Sigma_{o,o}$ are equal. Therefore, $o_1.A = o.A$ and $o_{k+1}.A = o.A$ implies that CPT^*_A is not an SPO which is a contradiction. □

We note that the order induced by a CP-net is an SPO provided that the corresponding conditional preference graph is acyclic. Otherwise, the SPO properties of the order induced by the CP-net are not guaranteed. In the case of HCP-nets, a conditional preference graph is always acyclic. As a result, the order induced by *any* HCP-net is an SPO.

EXAMPLE 5.3 *Let Mary have the following preference over cars. She unconditionally prefers newer cars, and likes GM more than Ford. If two cars are newer than* 2007*, she*

$$\begin{tikzcd}
\boxed{year} \arrow[d] \arrow[dr] & & \boxed{make} \arrow[dl] \\
\boxed{price} & \boxed{ml} \arrow[d] & \\
& \boxed{color} &
\end{tikzcd}$$

make
$o.make = GM \wedge o'.make = Ford$

year	*price*
> 2007	$o.price < o'.price$
≤ 2007	$o.price \leq 8000 \wedge o'.price > 8000$

year
$o.year > o'.year$

year	*make*	*ml*
> 2006	$GM, Ford$	$o.ml < o'.ml$
≤ 2006	GM	$o.ml < 60000 \wedge o'.ml \geq 60000$

ml	*color*
< 20000	$o.color = blue \wedge o'.color = red$
≥ 20000	$o.color = red \wedge o'.color = blue$

Figure 5-3: *HCP-net Υ_{N_3} from Example 5.3*

prefers the cheaper one. Otherwise, she does not want to spend more than 8000 *dollars on it. She prefers a car with less mileage in case of Fords and GMs newer than* 2006*. In case of GMs made before or in* 2006*, she prefers cars with mileage less than* 60000*. In case of the mileage less than* 20000*, she prefers a blue car to a red one. Otherwise, a red one is preferred to a blue one. An HCP-net N_3 representing these preferences is shown in Figure 5.3.*

5.2.1 Relationships with p-skyline framework

In Chapter 3, we introduced the p-skyline framework which generalizes the skyline framework by enriching it with the notion of *attribute importance*. In Section 3.4, we showed that every p-skyline relation is representable as a $(W, \mathcal{H})$-structure. Let us take a closer look at $(W, \mathcal{H})$-structures. The order $\succ_{(W,\mathcal{H})}$ induced by $(W, \mathcal{H})$ is defined as a composition of orders p_A for all attributes $A \in \mathcal{A}$. Semantically, p_A defines the way tuples are compared by the attribute A. In particular, o is preferred to o' according to p_A if 1) $o.A$ is preferred to $o'.A$ according to the atomic preference relation $>_A \in \mathcal{H}$, and 2) the values of the attributes $\mathcal{A} - (W_A \cup \{A\})$ are pairwise equal in o and o'. The values of W_A in o

and o' are not important. In the p-graph of the p-skyline relation which corresponds to the $(W, \mathcal{H})$-structure, the attribute W_A are the *children* of A.

Note that p_A for a $(W, \mathcal{H})$ is defined similarly to the order CPT_A^* induced by a conditional preference table $CPT_A = (\Phi_A, W_A, U_A)$ in the HCP-net framework. In particular, the values of W_A are also not important for a tuple o to be preferred to another tuple o' according to CPT_A^*. Moreover, W_A are the *descendants* of A in the conditional preference graph. Instead of the atomic preference relation $>_A$, the order R_q for some $q \in \Phi_A$ is used to compare the values of A. The values of the other attributes $Y_A = \mathcal{A} - (W_A \cup \{A\})$ should also be pairwise equal in o and o'.

In addition to that, the orders $\succ_{(W,\mathcal{H})}$ and $\succ_N$ induced by a $(W, \mathcal{H})$-structure and an HCP-net N are also defined similarly – as a transitive closure of the union of p_A and CPT_A^*, respectively.

However, despite all the similarities between p-skyline relations and HCP-nets, there are some important distinctions:

1. p-skyline relations lack the notion of *conditional importance*. Namely, given a p-skyline relation $\succ$ and an atomic preference relation $>_A$ used in construction of $\succ$, values of the attribute A are always compared using $>_A$ regardless of the values of the other attributes. At the same time, preferences over A in an HCP-net are conditioned on the values of attributes U_A;

2. the graphical representations of preferences in these model are semantically different. In the p-graph representing a p-skyline relation, an attribute has incoming edges from all more important attributes. In the conditional preference graphs of an HCP-net, an attribute has incoming edges from all the attributes its preferences is conditioned on. As a result, every p-graph is transitive which is not necessary the case for conditional preference graphs. Moreover, due to the requirement of using every atomic preference only once in the definition of a p-skyline relation, all p-graphs must satisfy the

`Envelope` property. This restriction does not apply to HCP-nets

5.3 Dominance testing for HCP-nets

In this section, we consider the problem of dominance testing for HCP-nets. We show here several methods of checking dominance. Before going to detail, let us introduce some notation to be used throughout the section.

DEFINITION 5.3 *Let N be an HCP-net and CPT_A be any of its conditional preference tables. Then for $q \in \Phi_A$, we define the order q^r as*

$$q^r = \{(o,o') \mid o.U_A = o'.U_A \in u_q, (o.A, o'.A) \in R_q\}.$$

The order CPT^r_A is defined as follows

$$CPT^r_A = \bigcup_{q \in \Phi_A} q^r$$

The relation q^r defined above is a *relaxed* version of the order q^* induced by the corresponding q. In contrast to q^*, in order for a tuple to be preferred to another tuple according to q^r, the values of the attributes Y_A in the tuples need not be pairwise equal. The same relationships holds between CPT^*_A and CPT^r_A. It is easy to check that q^* and CPT^r_A are SPO.

Definition 5.3 is illustrated in the next example. We use relaxed orders CPT^r_A in the method of dominance testing we propose further on.

EXAMPLE 5.4 *Consider the HCP-net N_3 from Example 5.3. Formulas representing the* *relations CPT^r for all conditional preference tables of N_3 are shown below.*

$$
\begin{aligned}
F_{CPT^r_{year}}(o,o') \equiv\ & o.year > o'.year \\
F_{CPT^r_{make}}(o,o') \equiv\ & o.make = GM \wedge o'.make = Ford \\
F_{CPT^r_{price}}(o,o') \equiv\ & o.year = o'.year \wedge (o.year > 2007 \wedge o.price < o'.price \vee \\
& o.year \leq 2007 \wedge o.price \leq 8000 \wedge o'.price > 8000) \\
F_{CPT^r_{ml}}(o,o') \equiv\ & o.year = o'.year \wedge o.make = o'.make \wedge \\
& (o.year > 2006 \wedge o.make \in \{GM, Ford\} \wedge o.ml < o'.ml \vee \\
& o.year \leq 2006 \wedge o.make = GM \wedge o.ml < 60000 \wedge o'.ml \geq 60000) \\
F_{CPT^r_{color}}(o,o') \equiv\ & o.ml = o'.ml \wedge (o.ml < 20000 \wedge o.color = blue \wedge o'.color = red \vee \\
& o.ml \geq 20000 \wedge o.color = red \wedge o'.color = blue)
\end{aligned}
$$

Now let us consider the dominance testing problem for HCP-nets. The first approach of testing if a tuple o is preferred to another tuple o' according to an HCP-net N is by applying Definition 5.2. One option here is to find a derivation sequence for $o \succ_N o'$ if it exists. We note that this problem is analogous to the *flipping sequence search* problem in the CP-net framework [BBD$^+$04]. However, in the HCP-net framework, this method may be impractical because orders induced by HCP-nets may be infinite. In such cases, materialization of orders is impossible. Another option is to compute the relation $\succ_N$ and then test $o \succ_N o'$. For finite relations, that problem is reduced to computing the orders CPT^*_A, taking their union and computing transitive closure of the result [CLRS01]. For finitely representable relations, the formula representation of $\succ_N$ has to be computed. The problem of computing such formulas is studied in Section 5.4.

Below we show two simple methods of testing dominance for HCP-nets which operate on conditional preference graphs.

THEOREM 5.2 *Let N be an HCP-net, and $o, o' \in \mathcal{U}$ be two different tuples. Let also* $\textbf{Diff}(o,o') \subseteq \mathcal{A}$ *be the set of the attributes in which o and o' are different, and* $\textbf{Top}(o,o')$ *be the set of the top-most attributes of* $\textbf{Diff}(o,o')$ *in Υ_N, i.e., those which have no ancestors in* $\textbf{Diff}(o,o')$*. Then the following are equivalent:*

1. $o \succ_N o'$;
2. $\forall A \in \textbf{Top}(o,o') \, . \, CPT^r_A(o,o')$
3. $\forall A \in \textbf{Diff}(o,o') \, \exists B \in Anc\text{-}self_{\Upsilon_N}(A) \, . \, CPT^r_B(o,o')$

PROOF

See Appendix A.

The dominance testing methods above intuitively follow from the fact that every attribute in a conditional preference graph is more important than its descendents. In particular, Method 2 in Theorem 5.2 implies that if we take all the attributes $\textbf{Diff}(o,o')$ in which two tuples o, o' are different, then o is preferred to o' if it is preferred to o' according to the most important attributes in $\textbf{Diff}(o,o')$. According to Method 3, o is preferred to o' if for every attribute in which o and o' are different, o is preferred to o' by this or a more important attribute. We note that similar dominance testing methods exist for p-skyline relations (Theorem 3.5).

EXAMPLE 5.5 *Consider the HCP-net N_3 from Example 5.3. Take two tuples* $t_1 = (GM,$ 2007, 10000, 40000, $blue)$ *and* $t_2 = (Ford, 2007, 6000, 40000, red)$ *describing cars with the attributes* $\mathcal{A} = \{make, year, price, ml, color\}$*. Then* $\textbf{Diff}(t_1,t_2) = \{make, price, color\}$*. Then* $\textbf{Top} = \{make, price\}$*,* $CPT^r_{make}(t_1,t_2)$*, and* $\neg CPT^r_{price}(t_1,t_2)$*. Hence,* $t_1 \not\succ_{N_3} t_2$*. Take* $t_3 = (GM, 2006, 6000, 40000, red)$*. Then* $\textbf{Diff}(o,o') = \{year, price, color\}$*,* $\textbf{Top} = \{year\}$*, and* $CPT^r_{year}(t_1,t_3)$*. Hence,* $t_1 \succ_{N_3} t_3$*.*

5.4 Formula representations of HCP-nets

In this section, we study the problem of constructing a formula representing the order induced by an HCP-net. This problem is important in the context of querying databases with HCP-nets. Recall that the winnow operator $w_{\succ}(r)$ in the binary relation framework is parameterized with the quantifier-free *formula* representing the order induced by the corresponding preference relation. In this section, we show how such a formula can be computed for the order induced by an HCP-net.

Take an HCP-net N. We assume here that for every attribute A and $q \in \Phi_A$, the relations u_q and R_q are representable using finite quantifier-free formulas F_{u_q} and F_{R_q}, correspondingly. Then the following formula represents CPT_A^r (Definition 5.3)

$$F_{CPT_A^r}(o,o') = F_{\approx_{U_A}}(o,o') \wedge \bigvee_{q \in \Phi_A} (F_{u_q}(o) \wedge F_{R_q}(o.A, o'.A)),$$

where $F_{\approx_{U_A}}$ is a formula representing $\approx_{U_A}$. Instances of such formulas are provided in Example 5.4. We note that $F_{CPT_A^r}$ is finite because $F_{\approx_{U_A}}$ is finite by construction, Φ_A is finite by definition, and we assumed that F_{u_q}, and F_{R_q} are finite. Moreover, the size of $F_{CPT_A^r}$ is $O(|\Phi_A| \cdot (S_{u_q} + S_{R_q}) + |U_A|)$, where $|\Phi_A|$ is the size of the set Φ_A, S_{u_q}, S_{R_q} are the maximum sizes of the formulas F_{u_q} and F_{R_q} among all $q \in \Phi_A$, and $|U_A|$ is the number of attributes in U_A.

The method of constructing a formula representing the order induced by an HCP-net is based on the last dominance test shown in Theorem 5.2.

PROPOSITION 5.2 *For an HCP-net N, the next formula represents the order induced by N*

$$F_{\succ_N}(o,o') \equiv \neg F_{\approx_{\mathcal{A}}}(o,o') \wedge \bigwedge_{A \in \mathcal{A}} \left(o.A = o'.A \vee \bigvee_{B \in Anc\text{-}self_{\Upsilon_N}(A)} CPT_B^r(o,o') \right)$$

The formula $F_{\succ_N}$ is constructed as follows. The first conjunct $\neg F_{\approx_{\mathcal{A}}}(o,o')$ is used to verify that the tuples o and o' are different. Then for every attribute $A \in \mathcal{A}$, we check if its values in o and o' are equal. If not, then $A \in \mathbf{Diff}(o,o')$. For every attribute in $\mathbf{Diff}(o,o')$, we check if o is preferred to o' according to CPT^r of any ancestor of A or A itself. If for some A, it is not the case, then o is not preferred to o' according to $\succ_N$, and it is otherwise. Hence, $F_{\succ_N}$ is a formula representation that uses the last dominance test from Theorem 5.2.

Note that another way of constructing a formula representation of an HCP-net induced order is by exploiting the first dominance test from Theorem 5.2. However, that requires enumeration of all possible sets **Top** for every attribute, and the number of such sets is exponential in general. As a result, the size of such a preference formula constructed this way may be exponential. In contrast, the size of a formula constructed according to Proposition 5.2 is polynomial.

PROPOSITION 5.3 *The size $|F_{\succ_N}|$ of $F_{\succ_N}$ defined in Proposition 5.2 is*

$$O(|\mathcal{A}|^3 + |\mathcal{A}|^2 \cdot S'_{\Phi} \cdot (S'_{u_q} + S'_{R_q})),$$

where $|\mathcal{A}|$ is the number of attributes in $\mathcal{A}$, S'_{Φ}, S'_{u_q}, and S'_{R_q} are the maximums of $|\Phi_A|$, S_{u_q}, and S_{R_q}, correspondingly, among all $A \in \mathcal{A}$.

PROOF

The expression for the formula size above can be easily derived from the expression of $F_{\succ_N}$ and using the facts that the size of CPT^r_A is $O(|\Phi_A| \cdot (S_{u_q} + S_{R_q}) + |U_A|)$, and the maximum of $|U_A|$ among all $A \in \mathcal{A}$ is $|\mathcal{A}|$. An HCP-net representable by the formula of size $\Theta(|\mathcal{A}|^3 + |\mathcal{A}|^2 \cdot S'_{\Phi} \cdot (S'_{u_q} + S'_{R_q}))$ has a conditional preference graph which is a total order of $\mathcal{A}$. □

EXAMPLE 5.6 *Consider the HCP-net N_3 from Example 5.3. Then the formula $F_{\succ_N}$ rep-*

resenting the order induced by N_3 is

$$\begin{aligned}
F_{\succ_{N_3}}(o,o') \equiv \neg(&o.make = o'.make \wedge o.year = o'.year \wedge o.price = o'.price \\
&\wedge o.ml = o'.ml \wedge o.color = o'.color) \\
&\wedge ((o.make = o'.make \vee F_{CPT^r_{make}}(o,o')) \\
&\wedge (o.year = o'.year \vee F_{CPT^r_{year}}(o,o')) \\
&\wedge (o.price = o'.price \vee F_{CPT^r_{price}}(o,o') \vee F_{CPT^r_{year}}(o,o')) \\
&\wedge (o.ml = o'.ml \vee F_{CPT^r_{ml}}(o,o') \vee F_{CPT^r_{year}}(o,o') \vee F_{CPT^r_{make}}(o,o')) \\
&\wedge (o.color = o'.color \vee F_{CPT^r_{color}}(o,o') \vee F_{CPT^r_{ml}}(o,o') \vee F_{CPT^r_{year}}(o,o') \\
&\quad \vee F_{CPT^r_{make}}(o,o')))
\end{aligned}$$

An important result implied by Proposition 5.3 is that preference relations induced by HCP-nets are representable by polynomial size formulas. In contrast, orders induced by CP-net are not known to have this property. We discuss some approaches of constructing formulas representing orders induced by CP-nets in Section 5.6.

5.5 Experimental evaluation

In this section, we present the results of the experimental evaluation of the HCP-net framework. Our goal was to compare methods of evaluating winnow queries in two frameworks: HCP-net and CP-net. We have implemented two algorithms for computing winnow queries with HCP-nets.

1. *Pairwise comparison (HCP-C)*: Given an HCP-net N and a relation r, we compute $w_{\succ_N}(r)$ by taking every pair $o, o' \in r$ and checking if o dominates o'. The result $w_{\succ_N}(r)$ is constructed as the set of all undominated tuples in r. To check if a tuple dominates another according to $\succ_N$, we used method 2 from Theorem 5.2.

2. *Database query (HCP-F)*: Given an HCP-net N and a database relation r, we compute $w_{\succ_N}(r)$ by evaluating the following preference query:

   ```
   SELECT * FROM r r1
   WHERE NOT EXISTS (SELECT * FROM r r2 WHERE F≻N(r2, r1))
   ```

   where the formula $F_{\succ_N}$ representing $\succ_N$ is computed according to Proposition 5.2.

To compute winnow for CP-nets, we used an approach analogous to *HCP-C*. We denote it as *CP-C*. In order to check if a tuple o dominates another tuple o' according to the order induced by a CP-net N', we implemented the technique of searching flipping sequences proposed in [BBD$^+$04]. To improve the performance of that algorithm, we implemented the *suffix fixing* heuristics discussed in the same paper. All algorithms used in the experiments were implemented in Java 6. We ran the experiments on Intel Core 2 Duo CPU 2.1 GHz with 2.0 GM RAM.

The experiments we carried out are divided into two groups. In the first group, we compared the performance of HCP-net and CP-net winnow queries on real life data. In the second group, we experimented with the scalability of HCP-net winnow queries. These experiments were carried out on synthetically generated datasets of anticorrelated tuples. The data used in all the experiments was stored in a PostgreSQL 8.3 database. The details of the experiments and their results are provided below.

Winnow queries: HCP-nets vs CP-nets

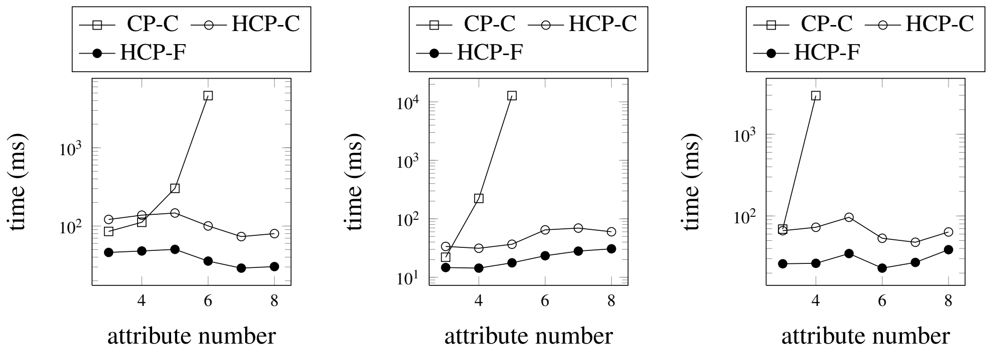
The aim of the experiments presented in this section is to demonstrate that the performance of winnow queries in the HCP-net framework is significantly higher than the one of CP-net winnow queries. Here we used the data set storing NHL player statistics [nhl08] of 2008. The data set contains 852 objects each of which having 18 attributes out of which we used at most 8.

We note that the running time of the algorithm for testing dominance for CP-nets we used here significantly depends on sizes of attribute domains. For the original attribute domain sizes (at most 239) and the CP-nets we used in the experiments, the dominance testing algorithm failed to terminate in reasonable time. To run the experiments, we transformed the dataset to reduce the domain sizes and made three different data sets each of which having 852 tuples whose attribute domains are of sizes 3, 4, and 5.

We picked 6 subsets of attributes of sizes 3 to 8. For every subset, we randomly generated 20 different conditional preference graphs with the corresponding set of nodes. For every node in a conditional preference graph, we generated a CPT with the number of rows at most 10. For every entry q, (i) R_q was generated as a total order of the values in the corresponding domain, and (ii) u_q was generated randomly. For every attribute, we tried to construct a set of R_q. The conditional preference graphs with the corresponding CPT were used to create 20 different CP-nets and HCP-nets. After that, we ran winnow algorithms for these nets. The results of our experiments are provided below.

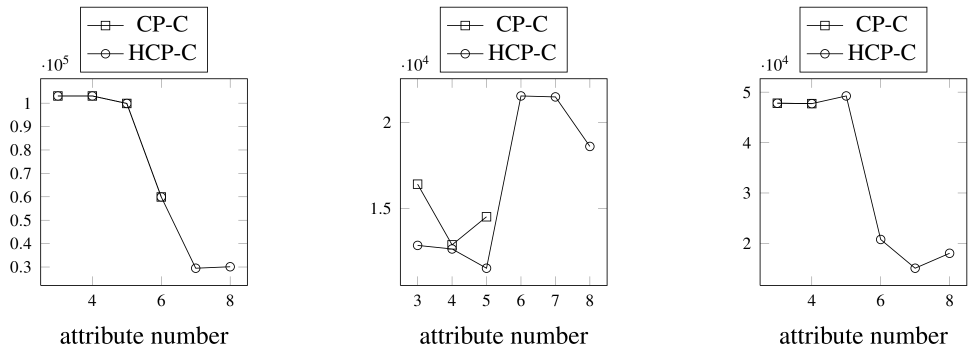
Figure 5-4 shows how the average running time of algorithms depends on the number of nodes in a conditional preference graph. In this experiments, we used the entire data set of 852 tuples. It can be observed that the time spent to evaluate winnow with a CP-net grows fast with the number of conditional preference graph nodes. It is explained by the fact that the flipping-sequence search space exponentially depends on number of nodes in conditional preference graph. At the same time, increasing the number of conditional preference graph nodes does not result in a significant performance drop in the case of HCP-nets. This is justified by the fact that the running times of HCP-C and HCP-F polynomially depend on the number of nodes in a conditional preference graph.

Figure 5-5 shows how the average number of pairwise tuple comparisons performed while computing winnow depends on the number of attributes in conditional preference graphs. This parameter is not available for HCP-F. We note that the curves for CP-C and



(a) domain size=3 (b) domain size=4 (c) domain size=5

Figure 5-4: *Winnow query performance vs conditional preference graph size*



(a) domain size=3 (b) domain size=4 (c) domain size=5

Figure 5-5: *Number of pairwise comparisons*

HCP-C almost always coincide. For the nets used in the experiments, the number of pairwise comparisons generally decreases with the size of conditional preference graph. This is due to the fact that increasing the number of attributes generally leads to enlarging the size of the corresponding preference relations. As a result, a tuple dominating another tuple is found faster. Note also that even though the number of comparisons decreases with the size of the conditional preference graph, the running time of CP-C goes up. This is due to the fact that it predominantly depends on the time needed to find a flipping sequence, which depends on the number of the attributes involved.

In Figure 5-6, we demonstrate how the winnow computation performance depends on data set size. Here we used nets with conditional preference graphs of 5 nodes. The do-

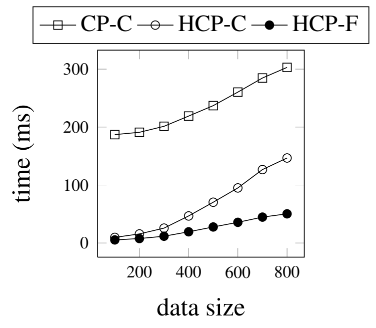


Figure 5-6: *Winnow query performance vs data size*

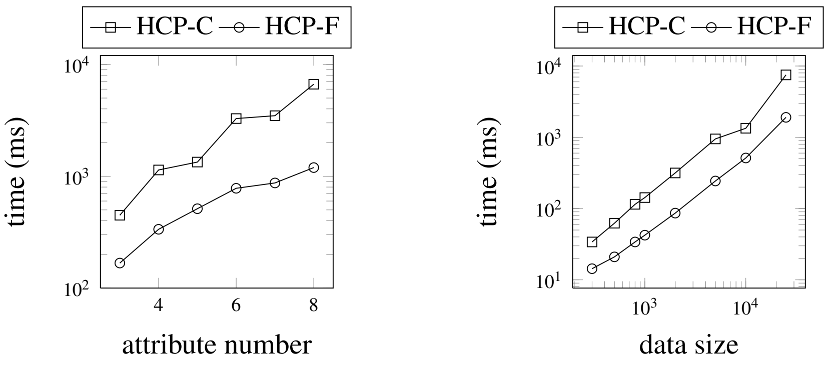
mains of all attributes were of size 3. When data set size goes up, the winnow computation performance drops for CP-nets and HCP-nets. This is justified by the fact that the running time of the winnow algorithms we used here is polynomial in data set size.

The results of these experiments show that the running times of the winnow computation algorithms for CP-net and HCP-net we used here are comparable only when the corresponding conditional preference graphs and attribute domains are small. If any of these parameters increases, then any HCP-net algorithm significantly outperforms the CP-net algorithm. Comparing the two HCP-net algorithms we used here – HCP-C and HCP-F – we can conclude that the preference-query based approach HCP-F is generally of higher performance.

Scalability of winnow query evaluation

In this set of experiments, we investigate the scalability of winnow query evaluation approaches in the HCP-net framework. Similarly to the previous section, we used 20 different randomly generated HCP-nets for sets of attributes of the size from 3 to 8. To run the experiments, we used a number of randomly generated data sets of sizes from 300 to 25,000 tuples. All tuples in these sets had 10 attributes. The attribute domains were integers in the range $[0,100)$.

In Figure 5-7(a), we plot the average time spent to evaluate winnow queries against



(a) Winnow performance vs conditional preference graph size

(b) Winnow performance vs data size

Figure 5-7: *Winnow query performance*

the number of attributes used in a conditional preference graph. The data set used in this experiment contained 10,000 tuples. The observed increase in the running time of both algorithms – HCP-C and HCP-F – is justified by the increase in the number of performed attribute comparisons in the former case and the increase in the preference formula size in the latter case. Figure 5-7(b) describes the dependence of the running time of the winnow algorithms on data set size. The HCP-nets used here had conditional preference graphs of 5 attributes. As one can observe, even when the data set contains 25,000 tuples, the winnow computation time does not exceed 10 seconds for HCP-C and 2 seconds for HCP-F.

Our experiments show that winnow queries in the HCP-net framework can be evaluated efficiently by pairwise comparison of tuples using the dominance testing method we proposed here as well as by evaluating SQL queries over a database table. According to our experiments, the performance of the HCP-net winnow algorithms does not significantly depend on sizes of attribute domains which is an important parameter for the CP-net algorithm we used. We have tested HCP-net query algorithms over data sets of relatively large size and observed that such queries may be evaluated in reasonable time. Another important observation is that evaluation of HCP-net winnow queries as database SQL queries is generally done faster than by performing pairwise dominance testing externally.

5.6 Related work

One of the main problems in the CP-net framework is dominance testing. In [BBD$^+$04], Boutilier et al defined that problem as searching for an *improving flipping sequence*. A flipping sequence is analogous to the notion of *derivation sequence* we use here. [BBD$^+$04] studied the dominance testing problem for certain subclasses of CP-nets. It was shown there that dominance testing is in `PTIME` for (i) binary-valued tree-structured CP-nets and (ii) binary-valued polytree CP-nets. It was also shown there that dominance testing is `NP`-complete for binary-valued direct-path singly connected CP-nets and `NP`-hard for acyclic CP-nets. [BBD$^+$04] introduced several general methods of testing dominance in the CP-net framework and proposed some optimization techniques. Another class of CP-nets for which dominance testing is in `PTIME` was shown by Yaman et al in [Yd07]. Goldsmith et al [GLTW05] showed that dominance testing for cyclic CP-nets is `PSPACE`-complete in general.

The problem of expressing variable relative importance of attributes in CP-net was addressed by Brafman et al in [BD02]. In that work, the TCP-net framework was proposed. That framework enhances CP-nets with two new notions: conditional and unconditional *relative attribute importance*. In addition to conditional preference edges, a TCP-net conditional preference graph has two types of edges: *i*-edges and *ci*-edges. An *i*-edge going from an attribute A to another attribute B represents the fact that A is unconditionally more important than B. A *ci*-edge going from A to B is labeled with a set of pairs of (*condition*, *importance direction*), where *condition* is an assignment to any subset of $\mathcal{A} - \{X, Y\}$, and *importance direction* defines if A is more important than B or vice versa for the condition. [BD02] showed that if the graph which is a union of the conditional preference graph, *i*-, and *ci*-edges is acyclic, then the corresponding order induced by the TCP-net is an SPO. Similarly to CP-nets, dominance testing here is defined as existence of an improving flipping sequence. An extension of TCP-nets was proposed by Dimopoulos et al in [DMT05].

That work addresses the limitation of the original TCP-nets that a variable cannot be more important than its ancestors in a conditional preference graph. It also illustrates some cases when cyclic TCP-nets may induce SPO relations.

Another framework of extending CP-nets with attribute importance has been proposed by Wilson in [Wil04]. Similarly to the HCP-net framework, every attribute A is associated with a set of "don't care" attributes W_A, i.e., those which are not taken into account when comparing tuples by that attribute. Moreover, such nets are equivalent to CP-nets when W_A is empty for every $A \in \mathcal{A}$. That framework is more general than the HCP-framework we have introduced here, because the sets W_A are not restricted to the descendents of A. However, [Wil04] does not consider how to deal with infinite domain attributes, which are essential for the binary relation preference framework. [Wil04] proves that the order induced by some classes of the extended CP-networks are strict partial orders. However, it does not address the problem of computing preference formulas representing such orders. In [Wil06], Wilson uses that CP-net extension to construct orders approximating orders induced by CP-nets. He also shows that dominance testing for such extensions may be performed in polynomial time.

The problem of representing orders induced by (T)CP-nets as formulas has been addressed by Brafman et al in [BD04]. That work proposed a method of computing a utility function *approximating* the order induced by a net. In particular, such a function represents a weak order compatible with the SPO induced by the network. However, no complexity analysis of the algorithm running time is provided in this work. An attempt to construct a quantifier-free first order formula representing the preference relation induced by a CP-net has been proposed by Chomicki in [CS05]. It was developed further and adopted to TCP-nets by Endres et al in [EK06]. The main idea of this method is to construct formulas F_{CPT^*} representing the relations induced by all CPT of a TCP-network. After that, a formula representing the order induced by the net is constructed as the *transitive closure* of the union of the corresponding relations *CPT^**. The transitive closure is computed using *Constraint Datalog*. When Constraint Datalog is used to compute transitive closure, one needs to show that the evaluation terminates (which is the case only for some constraint theories [KKR95]). Moreover, one typically obtains at best exponential bounds on the resulting formula size. In contrast to that, the method of constructing preference formulas for HCP-nets we have proposed here results in polynomial size formulas. [EK06] also gives an example of embedding preferences over infinite domain attributes into TCP-networks. Namely, it shows how one can represent a CPT for an infinite domain attribute using a limited set of preference constructors for which the transitive closure Datalog program terminates. In contrast, in our work we have no limitations to the class of SPO preferences formulas used in CPT representation.

In [MC07], we presented an algorithm for computing preference formulas representing the order induced by HCP-nets. The algorithm decomposes the conditional preference graph of an HCP-net into *subnets* and of *many-to-one* connectives. After that, formulas representing all subnets and connectives are computed and aggregated into a single formula using *Pareto* and *prioritized* accumulation operators [Kie04]. We showed that the formulas computed by the algorithm are of exponential size in general and of polynomial size for a certain class of HCP-nets.

Chapter 6

Preference contraction

In this chapter, we propose the operation of preference contraction in the binary relation preference framework.

6.1 Requirements to preference contraction

Preference contraction is an operation of changing a preference relation by discarding its subset. Preferences are generally discarded if the reasons for holding them are no longer valid. It was argued that discarding preferences is one of the fundamental operations of changing preferences [Han95].

EXAMPLE 6.1 *Assume that Mary wants to buy a car. She prefers newer cars, and given two cars made in the same year, a cheaper one is preferred. This preference relation is defined by the formula*

$$o \succ o' \equiv o.year > o'.year \vee o.year = o'.year \wedge o.price < o'.price,$$

where the attribute year defines the year when cars are made, and the attribute price - their price. The information about all cars which are in stock now is shown in the table below:

id	*make*	*year*	*price*
t_1	*vw*	*2007*	*15000*
t_2	*bmw*	*2007*	*20000*
t_3	*kia*	*2006*	*15000*
t_4	*kia*	*2007*	*12000*

Then the set of the most preferred cars according to $\succ$ is $S_1 = \{t_4\}$. Assume that having observed the set S_1, Mary understands that it is too narrow. She decides that the car t_1 is not really worse than t_4. She generalizes that by stating that the cars made in 2007 *which cost* 12000 *are not better than the cars made in* 2007 *costing* 15000*. So t_4 is not preferred to t_1 any more, and thus the set of the best cars according to the new preference relation should be $S_3 = \{t_1, t_4\}$.*

The problem which we face here is how to change the preference relation $\succ$ accordingly.

As a preference change operation, preference contraction has to satisfy some typical properties of operations of change. First, it was shown in [Doy04] that along with the discovery of sources of preference change and elicitation of the change itself, it is important to preserve the *correctness of preference model* in the presence of change. In the binary relation framework, a natural correctness criterion is the preservation of SPO properties of preference relations. Indeed, if an original set of preferences form a valid preference relation, it is intuitive to expect that the contracted set of preferences will also form a valid preference relation.

Another fundamental property of a preference change operation is the *minimality of change*. It supports the intuition that in order to incorporate a certain change to a preference relation, the relation itself should not be changed more than necessary to successfully perform the operation. In the case of the preference contraction discussed here, such minimality is measured in terms of set-theoretic inclusion.

We illustrate these properties in the next example.

EXAMPLE 6.2 *Pick the preference change discussed in Example 6.1 and incorporate it to the preference relation $\succ$. In particular, we want to find a preference relation obtained from*

$\succ$, in which certain preferences do not hold. A naive solution is to represent the new preference as $\succ_1 \equiv (\succ - CON)$, where $CON(o,o') \equiv o.year = o'.year = 2007 \wedge o.price = 12000 \wedge o'.price = 15000$, i.e., CON is the preference we want to discard. So

$$\begin{aligned} o \succ_1 o' \equiv &(o.year > o'.year \vee o.year = o'.year \wedge o.price < o'.price) \wedge \\ &\neg(o.year = o'.year = 2007 \wedge o.price = 12000 \wedge o'.price = 15000). \end{aligned}$$

However, $\succ_1$ is not transitive since if we take $t_5 = (bmw, 2007, 12000)$, $t_6 = (bmw,$ 2007, 14000$)$, and $t_7 = (bmw, 2007, 15000)$, then $t_5 \succ_1 t_6$ and $t_6 \succ_1 t_7$ but $t_5 \not\succ_1 t_7$. Hence, this change does not preserve SPO. To make the changed preference relation transitive, some other preferences have to be discarded in addition to CON. At the same time, discarding too many preferences is not a good solution since they may be important. Therefore, we need to discard a minimal part of $\succ_1$ which contains CON and preserves SPO in the modified preference relation. An SPO preference relation which is minimally different from $\succ_1$ and does not contain CON is shown below:

$$\begin{aligned} o \succ_2 o' \equiv &(o.y > o'.y \vee o.y = o'.y \wedge o.p < o'.p) \wedge \\ &\neg(o.y = o'.y = 2007 \wedge o.p = 12000 \wedge o'.p > 12000 \wedge o'.p \leq 15000) \end{aligned}$$

The set of the best cars according to $\succ_2$ is $S_2' = \{t_1, t_4\}$. As we can see, the relation $\succ_2$ is different from the naive solution $\succ_1$ in the sense that $\succ_2$ implies that a car made in 2007 *costing* 12000 *is not better than a car made in* 2007 *costing* from 12000 to 15000.

The example above shows that to discard the subset *CON* (called the *base contractor* here) of the preference relation $\succ$, some preferences additional to *CON* may be discarded to make the resulting preference relation an SPO. A subset P^- of $\succ$ which containts *CON* and whose removal from $\succ$ preserves the SPO axioms of the modified preference relation is called a *full contractor of $\succ$ by CON*.

6.2 Scenario of minimal preference contraction

In this section, we consider possible scenarios of user guided preference contraction. For that, let us consider the notion of *full contractor* informally defined above. A full contractor P^- may be viewed as a union of the preferences CON to discard and *a set of reasons* of discarding CON. Ideally, if a user decides to discard preferences, she also provides all the reasons for such a change. In this case, the relation $(\succ - CON)$ is a consistent preference relation (i.e., SPO). However, in real life scenarios, it is hard to expect that users always provide complete information about the change they want to make. At the same time, the number of alternative full contractors P^- for a given $\succ$ and CON may be large or even infinite for infinite preference relations. As a result, there is often a need to learn from the user the reasons for discarding preferences. That may be done in a step-wise manner by exploring possible alternatives and using user feedback to select the correct ones.

We envision the following scenario here. To find a complete set of preferences she wants to discard, the user iteratively expresses the most obvious preferences CON that should be dropped from her preference relation $\succ$. After that, a possible set of reasons P^- for such a change is computed. To check if she is satisfied with the computed P^-, the impact of the performed change may be demonstrated to her (e.g., the result of the winnow operator over a certain data set). If the full contractor P^- does not represent the change she actually wanted to make, the user may undo the change and select another alternative or tune the contraction by elaborating it. One type of such elaboration can be expressed as a set of additional preferences to discard. Another type of elaboration which we propose here is *preference protection*. Because the exact reasons for contracting CON are not known beforehand, some preferences which are important for the user may be contracted in an intermediate P^-. To avoid that, a user can impose a requirement of *protecting a set of preferences from removal*. The corresponding contraction operator is called here *preference-protecting contraction*. This iterative process stops when the user is satisfied with the computed full contractor.

An important property of the scenario above is that the set of reasons P^- which is computed as a result of the iterative process above has to be as small as possible. That is, preferences that the user does not want to discard and that are not needed to be removed to preserve the consistency of the modified preference relation should remain. To preserve the minimality of preference change, two approaches are possible.

In the first one, the full contractor computed in every step is *minimal*. The corresponding contraction operator here is called *minimal preference contraction*. It is guaranteed that the full contractor P^- computed in the last iteration (i.e., when P^- is satisfactory for the user) is minimal. Note that since there could be many possible minimal full contractors of a preference relation by a base contractor, *any* of them may be picked assuming that if the user is not completely satisfied with it, she will tune the contraction in the next iteration. In belief revision theory, the contraction operator with a similar semantics is called *maxichoice contraction* [Han98].

In the other variant, the full contractor P^- computed in every step is not necessary minimal. However, the user can make P^- smaller by specifying the preferences which should be protected from removal. The full contractor computed in the last step may be not minimal, but sufficiently small to meet the user expectations. We propose to construct P^- as the *union* of all minimal full contractors of the preference relation by CON. This contraction operator is called *meet contraction* if no preferences need to be preserved, and *preference-protecting meet contraction* if preference preservation is required. Similar operators in belief revision are full meet contraction, and partial meet contraction [Han98].

We note that the operations of preference contraction we propose in this chapter should be understood in the context of the scenario discussed above. However, the practical details of the scenario are beyond the scope of this work.

6.3 Preference contraction in binary relation framework

Here we formally define the operation of preference contraction in the binary relation framework. We assume that when a user intends to discard some preferences, he or she expresses the preferences to discard as a binary relation called a *base contractor*. The interpretation of each pair in a base contractor is that the first tuple should not be preferred to the second tuple. We require base contractor relations to be subsets of the preference relation to be contracted. Hence, a base contractor is irreflexive but not necessary transitive. Apart from that, we do not impose any other restrictions on the base contractors (e.g., they can be finite of infinite), unless stated otherwise. Throughout the chapter, base contractors are typically referred to as *CON*.

DEFINITION 6.1 *A binary relation P^- is* a full contractor of a preference relation $\succ$ by *CON if $CON \subseteq P^- \subseteq \succ$, and $(\succ - P^-)$ is a preference relation (i.e., an SPO). The relation $(\succ - P^-)$ is called the* contracted relation.

A relation P^- is a minimal full contractor of $\succ$ by *CON if P^- is a full contractor of $\succ$ by CON, and there is no other full contractor P' of $\succ$ by CON s.t. $P' \subset P^-$.*

DEFINITION 6.2 *A preference relation is* minimally contracted *if it is contracted by a* minimal full contractor. Contraction *is an operation of constructing a full contractor.* Minimal contraction *is an operation of constructing a minimal full contractor.*

The notion of a minimal full contractor narrows the set of full contractors. However, as we illustrate in Example 6.3, a minimal full contractor is generally not unique for the given preference and base contractor relations. Moreover, the number of minimal full contractors for infinite preference relations can be infinite. Thus, minimal contraction differs from minimal preference revision [Cho07b] which is uniquely defined for given preference and revising relations.

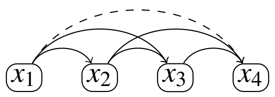


Figure 6-1: *$\succ$ and CON*

EXAMPLE 6.3 *Take the preference relation $\succ$ which is a total order of $\{x_1,\ldots,x_4\}$ (Figure 6-1). Let the base contractor relation CON be $\{x_1x_4\}$. Then the following sets are minimal full contractors of $\succ$ by CON: $P_1^- = \{x_1x_2, x_1x_3, x_1x_4\}$, $P_2^- = \{x_3x_4, x_2x_4, x_1x_4\}$, $P_3 = \{x_1x_2, x_3x_4, x_1x_4\}$, and $P_4^- = \{x_1x_3, x_2x_4, x_2x_3, x_1x_4\}$.*

An important observation here is that that the contracted preference relation is defined as a subset of the original preference relation. We want to preserve the SPO properties – transitivity and irreflexivity – of preference relations. Since any subset of an irreflexive relation is also an irreflexive relation, no additional actions are needed to preserve irreflexivity during contraction. However, not every subset of a transitive relation is a transitive relation. We need to consider paths in the original preference relation which by transitivity may produce *CON*-edges which need to be discarded. We call such paths *CON-detours*.

DEFINITION 6.3 *Let $\succ$ be a preference relation, and $P \subseteq \succ$. Then a $\succ$-path from x to y is a P-*detour* if $xy \in P$.*

6.4 Properties of full contractors

First, let us consider the problem of finding any full contractor, not necessary minimal. As we showed above, a contracted preference relation cannot have any *CON*-detours. To achieve that, some additional edges of the preference relation have to be discarded. However, when we discard these edges, we have to make sure that there are no paths in the contracted preference relation which produce the removed edges. Hence, a necessary and sufficient condition for a subset of a preference relation *to be its full contractor* can be formulated in an intuitive way.

LEMMA 6.1 *Given a preference relation (i.e., an SPO) $\succ$ and a full contractor CON, a relation $P^- \subseteq \succ$ is a full contractor of $\succ$ by CON if and only if $CON \subseteq P^-$, and for every $xy \in P^-$, $(\succ - P^-)$ contains no paths from x to y.*
PROOF

$\boxed{\Leftarrow}$ Prove that if for all $xy \in P^-$, $(\succ - P^-)$ contains no paths from x to y, then $(\succ - P^-)$ is an SPO. The irreflexivity of $(\succ - P^-)$ follows from the irreflexivity of $\succ$. Assume $(\succ - P^-)$ is not transitive, i.e., there are $xz, zy \in (\succ - P^-)$ but $xy \notin (\succ - P^-)$. If $xy \in P^-$ then the path xz, zy is not disconnected which contradicts the initial assumption. If $xy \notin P^-$, then the assumption of transitivity of $\succ$ is violated.
$\boxed{\Rightarrow}$ First, $CON \not\subseteq P^-$ implies that P^- is not a full contractor of $\succ$ by CON by definition. Second, the existence of a path from x to y in $(\succ - P^-)$ for $xy \in P^-$ implies that $(\succ - P^-)$ is not transitive, which violates the SPO properties. □

Now let us consider the property of minimality of full contractors. Let P^- be any minimal full contractor of a preference relation $\succ$ by a base contractor CON. Pick any edge xy of P^-. An important question which arises here is *why is xy a member of P^-*? The answer is obvious if xy is also a member of CON: every CON-edge has to be removed from the preference relation. However, what if xy is not a member of CON? To answer this question, let us introduce the notion of the *outer edge set* of an edge belonging to a full contractor relation.

DEFINITION 6.4 *Let CON be a base contractor of a preference relation $\succ$, and P^- be a full contractor of $\succ$ by CON. Let $xy \in P^- - CON$, and*

$$\Phi_0(xy) = \{xy\}, \textit{ and}$$

$$\Phi_i(xy) = \{u_i v_i \in P^- | \exists u_{i-1} v_{i-1} \in \Phi_{i-1}(xy) \,.\, u_i = u_{i-1} \wedge v_{i-1} v_i \in (\succ - P^-) \vee v_{i-1} = v_i \wedge u_i u_{i-1} \in (\succ - P^-)\}, \textit{ for } i > 0.$$

Then the outer edge set $\Phi(xy)$ for xy *is defined as*

$$\Phi(xy) = \bigcup_{i=0}^{\infty} \Phi_i(xy).$$

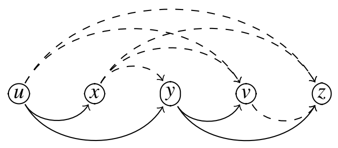


Figure 6-2: $\Phi(xy)$ *for Example 6.4.*

Intuitively, the outer edge set $\Phi(xy)$ of an edge $xy \in (P^- - CON)$ contains all the edges of a full contractor P^- which should be removed from P^- (i.e., added back to the preference relation $\succ$) to preserve the full contractor property of the result, should xy be removed from P^- (i.e., added back to the preference relation). The reasoning here is as follows. When for some i, $\Phi_i(xy)$ is removed from P^-, then $\Phi_{i+1}(xy)$ has to be also removed from P^-. Otherwise, for every edge in $\Phi_{i+1}(xy)$, there is a two-edge path in $\succ$ one of whose edges is in $\Phi_i(xy)$ while the other is not contracted. Hence, if the SPO properties of $(\succ - P^-)$ need to be preserved, removing xy from P^- requires recursively removing the entire $\Phi(xy)$ from P^-.

The next example illustrates the inductive construction of an outer edge set. Some properties of outer edge sets are shown in Lemma 6.2.

EXAMPLE 6.4 *Let a preference relation $\succ$ be the set of all edges in Figure 6-2, and P^- be defined by the dashed edges. Let us construct $\Phi(xy)$ (assuming that xy is not an edge of the base contractor CON).*

- $\Phi_0(xy) = \{xy\}$*;*
- $\Phi_1(xy) = \{xv, xz\}$*;*

- $\Phi_2(xy) = \{uv, uz\}$;

Thus, $\Phi(xy) = \{xy, xv, xz, uv, uz\}$.

LEMMA 6.2 *Let P^- be a full contractor of a preference relation $\succ$ by a base contractor CON. Then for every $xy \in (P^- - CON)$, $\Phi(xy)$ has the following properties:*

1. *for all $uv \in \Phi(xy)$, $u \succeq x$ and $y \succeq v$;*
2. *for all $uv \in \Phi(xy)$, $ux, yv \notin P^-$;*
3. *if $(\Phi(xy) \cap CON) = \emptyset$, then $P' = (P^- - \Phi(xy))$ is a full contractor of $\succ$ by CON.*

PROOF

First, we prove that Properties 1 and 2 hold. We do it by induction on the index of $\Phi_i(xy)$ used to construct $\Phi(xy)$. For every $uv \in \Phi_0(xy)$, Properties 1 and 2 hold by the construction of Φ_0. Now let Properties 1 and 2 hold for $\Phi_n(xy)$, i.e.,

$$\forall u_n v_n \in \Phi_n(xy) \rightarrow u_n \succeq x \wedge y \succeq v_n \wedge u_n x, y v_n \notin P^- \tag{1}$$

Pick any $u_{n+1}v_{n+1} \in \Phi_{n+1}(xy)$. By construction of $\Phi_{n+1}(xy)$, we have

$$\begin{aligned} \exists u_n v_n \in \Phi_i(xy) \ . \ u_{n+1} = u_n \wedge v_n \succ v_{n+1} \wedge v_n v_{n+1} \notin P^- \vee \\ u_{n+1} \succ u_n \wedge v_n = v_{n+1} \wedge u_{n+1} u_n \notin P^- \end{aligned} \tag{2}$$

Note that $u_{n+1} \succeq x$ and $y \succeq v_{n+1}$ follows from (1), (2), and transitivity of $\succeq$. Similarly, $u_{n+1}x, yv_{n+1} \notin P^-$ is implied by (1), (2), and transitivity of $(\succ - P^-)$. Hence, Properties 1 and 2 hold for $\cup_{i=0}^{n} \Phi_i(xy)$ for any n.

Now we prove Property 3: $(\succ - P')$ is an SPO and $CON \subseteq P'$. The latter follows from $CON \subseteq P^-$ and $\Phi(xy) \cap CON = \emptyset$. Irreflexivity of $(\succ - P')$ follows from irreflexivity of $\succ$. Assume $(\succ - P')$ is not transitive, i.e., there are $uv \notin (\succ - P')$ and $uz, zv \in (\succ - P')$.

Transitivity of $(\succ - P^-)$ implies that at least one of uz, zv is in $\Phi(xy)$. However, Property 1 implies that *exactly one* of uz, zv is in $\Phi(xy)$ and the other one is not in $\Phi(xy)$ and thus in $(\succ - P^-)$. However, $uz \in \Phi(xy)$ and $zv \in (\succ - P^-)$ imply $uv \in \Phi(xy)$, and thus $uv \in (\succ - (P^- - \Phi(xy))) = (\succ - P')$, i.e., we derive a contradiction. A similar contradiction is derived in the case $uz \in (\succ - P^-)$ and $zv \in \Phi(xy)$. Therefore, $(\succ - P')$ is an SPO and P' is a full contractor of $\succ$ by CON. $\square$

Out of the three properties shown in Lemma 6.2, the last one is the most important. It says that if an edge xy of a full contractor is not needed to disconnect any CON-detours, then that edge may be dropped from the full contractor along with its entire outer edge set. A more general result which follows from Lemma 6.2 is formulated in the next theorem. It represents a necessary and sufficient condition for a full contractor to be *minimal*.

THEOREM 6.1 **(Full-contractor minimality test).** *Let P^- be a full contractor of $\succ$ by CON. Then P^- is a* minimal *full contractor of $\succ$ by CON if and only if for every $xy \in P^-$, there is a CON-detour in $\succ$ in which xy is the only P^--edge.*

PROOF

$\boxed{\Leftarrow}$ The proof in this direction is straightforward. Assume that for every edge of the full contractor P^- there exists at least one CON-detour in which only that edge is in P^-. If P^- loses any its subset P containing that edge, then there will be a CON-detour in $\succ$ having no edges in $(P^- - P)$, and thus $(P^- - P)$ is not a full contractor of $\succ$ by CON by Lemma 6.1. Hence, P^- is a minimal full contractor.

$\boxed{\Rightarrow}$ Let P^- be a minimal full contractor. For the sake of contradiction, assume for some $xy \in P^-$, 1) there is no CON-detour which xy belongs to, or 2) any CON-detour xy belongs to has at least one more P^--edge. If 1) holds, then $\Phi(xy)$ has no edges in CON by construction. Thus, Lemma 6.2 implies that $(P^- - \Phi(xy))$ is a full contractor of $\succ$ by CON. Since

$\Phi(xy)$ is not empty, we get that P^- is not a minimal full contractor which is a contradiction. If 2) holds, then we use the same argument as above and show that $\Phi(xy) \cap CON = \emptyset$. If $\Phi(xy) \cap CON$ is not empty (i.e., some $uv \in \Phi(xy) \cap CON$), then by Lemma 6.2,

$$u \succeq x \wedge x \succ y \wedge y \succeq v \wedge ux, yv \notin P^-,$$

and thus there is a CON-detour going from u to v in which xy is the only P^--edge. This contradicts the initial assumption. □

Note that using the definition of minimal full contractor to check the minimality of a full contractor P^- requires checking the full contractor properties of *all subsets* of P^-. In contrast, the minimality checking method shown in Theorem 6.1 requires checking properties of *distinct elements* of P^- with respect to its other members.

Sometimes a direct application of the minimality test from Theorem 6.1 is hard because it does not give any bound on the length of CON-detours. Hence, it is not clear how it can be represented as a finite formula. Fortunately, the transitivity of preference relations implies that the minimality condition from Theorem 6.1 can be stated in terms of paths of length at most three.

COROLLARY 6.1 *A full contractor P^- of $\succ$ by CON is* minimal *if and only if for every edge $xy \in P^-$, there is a CON-detour consisting of at most three edges among which only xy is in P^-.*

PROOF

$\boxed{\Leftarrow}$ Trivial.

$\boxed{\Rightarrow}$ For every $xy \in P^-$, pick any CON-detour T in which the only P^--edge is xy. If its length is less or equal to three, then the corollary holds. Otherwise, x is not the start node of T, or y is not the end node of T, or both. Let the start node u of T be different from

x. Since the only common edge of T and P^- is xy, every edge in the path from u to x is an element of $(\succ - P^-)$. Transitivity of $(\succ - P^-)$ implies $ux \in (\succ - P^-)$. Similarly, $yv \in (\succ - P^-)$ for the end node of T if y is different from v. Hence, there is a CON-detour of length at most three in which xy is the only element of P^-. □

As a result, the following tests can be used to check the minimality of a full contractor P^-. In the finite case, P^- is minimal if the following relational algebra expression results in an empty set

$$\begin{aligned}\mathrm{P} - [&\pi_{\mathrm{P_2.X,P_2.Y}}((\mathrm{R_1 - P_1}) \underset{\mathrm{R_1.Y=P_2.X}}{\bowtie} \mathrm{P_2} \underset{\mathrm{P_2.Y=R_3.X}}{\bowtie} (\mathrm{R_3 - P_3}) \underset{\mathrm{R_1.X}=C.X,\ \mathrm{R_3.Y=C.Y}}{\bowtie} \mathrm{C}) \cup \\ &\pi_{\mathrm{P_2.X,P_2.Y}}(\mathrm{P_2} \underset{\mathrm{P_2.Y=R_3.X}}{\bowtie} (\mathrm{R_3 - P_3}) \underset{\mathrm{P_2.X}=C.X,\ \mathrm{R_3.Y=C.Y}}{\bowtie} \mathrm{C}) \cup \\ &\pi_{\mathrm{P_2.X,P_2.Y}}((\mathrm{R_1 - P_1}) \underset{\mathrm{R_1.Y=P_2.X}}{\bowtie} \mathrm{P_2} \underset{\mathrm{R_1.X}=C.X,\ \mathrm{P_2.Y=C.Y}}{\bowtie} \mathrm{C}) \cup \mathrm{C}\,],\end{aligned}$$

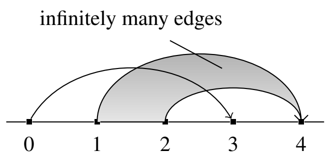
for the tables R, C and P with columns X and Y, storing $\succ$, CON, and P^- correspondingly. In the finitely representable case, P^- is minimal if the following formula is valid

$$\begin{aligned}\forall x,y\ (F_{P^-}(x,y) \Rightarrow F_{\succ}(x,y) \wedge \exists u,v\ .\ F_{CON}(u,v) \wedge (F_{\succ}(u,x) \vee u = x) \wedge \\ (F_{\succ}(y,v) \vee y = v) \wedge \neg F_{P^-}(u,x) \wedge \neg F_{P^-}(y,v)).\end{aligned}$$

Below we show examples of checking minimality of full contractors using Corollary 6.1. We note that when the relations are definable using ERO-formulas, checking minimality of a full contractor can be done by performing quantifier elimination on the above formula.

EXAMPLE 6.5 *Let a preference relation $\succ$ be defined by the formula $F_{\succ}(o,o') \equiv o.d < o'.d$, where d is a $\mathcal{Q}$-attribute. Let a base contractor CON of $\succ$ be defined by the formula*

$$F_{CON}(o,o') \equiv (1 \le o.d \le 2 \wedge o'.d = 4) \vee (o.d = 0 \wedge o'.d = 3)$$



(a) Infinite case, Example 6.5

R	X	Y
	u	x
	x	y
	y	v
	u	y
	x	v
	u	v

C	X	Y
	u	v

P	X	Y
	u	x
	y	v
	x	v
	u	v

D	X	Y
	u	x
	x	v

(b) Finite case, Example 6.6

Figure 6-3: *Checking minimality of a full contractor*

(Figure 6-3(a)). Denote the relation represented by the first and second disjuncts of F_{CON} as CON_1 and CON_2 correspondingly. The relation P^- defined by F_{P^-} is a full contractor of $\succ$ by CON

$$F_{P^-}(o,o') \equiv (1 \le o.d \le 2 \wedge 2 < o'.d \le 4) \vee (o.d = 0 \wedge 0 < o.d' \le 3).$$

Similarly, denote the relations represented by the first and the second disjuncts of F_{P^-} as P_1^- and P_2^- correspondingly. We use Corollary 6.1 to check the minimality of P^-. By the corollary, we need to consider CON-detours of length at most three. Note that every P_1^--edge starts a one- or two-edge CON-detour with the corresponding CON_1-edge. Moreover, the second edge of all such two-edge detours is not contracted by P^-. Hence, the minimal full contractor test is satisfied for P_1^--edges. Now we consider P_2^--edges. All CON-detours which these edges belong to 1) correspond to CON_2-edges, and 2) are started by P_2^--edges. Hence, we need to consider only CON_2-detours of length at most two. When a P_2^--edge ends in o' with the value of d in $(0,1)$ and $(2,3]$, the second edge in the corresponding two-edge CON_2-detour is not contracted by P^-. However, when d is in $[1,2]$, the second edge is already in P^-. Hence, P^- is not minimal by Corollary 6.1. To minimize it, we construct P^* by removing the edges from P^- which end in o' with d in $[1,2]$

$$F_{P^*}(o,o') \equiv (1 \le o.d \le 2 \wedge 2 < o'.d \le 4) \vee (o.d = 0 \wedge (0 < o.d' < 1 \vee 2 < o'.d \le 3))$$

EXAMPLE 6.6 *Take a preference represented by the table R, and a base contractor R represented by the table C (Figure 6-3(b)). Consider the table P representing a base contractor of R by C. Then the result of the relational algebra expression above evaluated for these tables is shown in the table D. Since it is not empty, the full contractor represented by P is not minimal. The minimality of P can be achieved by removing from it any (but only one) tuple in D.*

6.5 Construction of a minimal full contractor

In this section, we propose a method of computing a minimal full contractor. We use the idea shown in Example 6.3. Pick for instance the set P_1^-. That set was constructed as follows: we took the CON-edge x_1x_4 and put in P_1^- all the edges which start some path from x_1 to x_4. For the preference relation $\succ$ from Example 6.3, P_1^- turned out to be a minimal full contractor. As is it shown in the next lemma, the set consisting of all edges starting CON-detours is a full contractor by CON.

LEMMA 6.3 *Let $\succ$ be a preference relation and CON be a base contractor relation of $\succ$. Then*

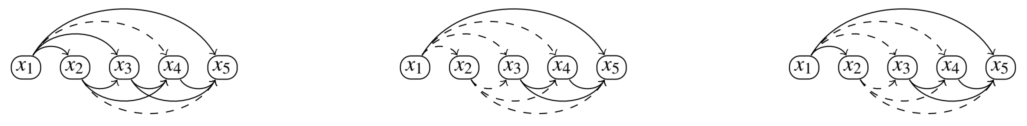
$$P^- := \{\, xy \mid \exists x'v \in CON \,.\, x' = x \wedge x' \succ y \wedge y \succeq v\}$$

is a full contractor of $\succ$ by CON.

PROOF

By construction of P^-, $CON \subseteq P^-$. Lemma 6.1 implies $(\succ - P^-)$ is an SPO. Hence, $(\succ - P^-)$ is a full contractor of $\succ$ by CON. □

However, in the next example we show that such a full contractor is not always minimal. Recall that by Theorem 6.1, for every edge of a full contractor there should be a CON-detour which only shares that edge with the contractor. However, it may be the case that an edge starting a *CON*-detour does not have to be discarded because the *CON*-detour is already disconnected.



(a) Preference relation $\succ$ (b) Preference relation $(\succ - P^-)$ (c) Minimally contracted $\succ$

Figure 6-4: *Preference contraction*

EXAMPLE 6.7 *Let a preference relation* $\succ$ *be a total order of* $\{x_1,\ldots,x_5\}$ *(Figure 6-4(a)). Let a base contractor CON be* $\{x_1x_4,x_2x_5\}$*. Let* P^- *be defined as in Lemma 6.3. That is* $P^- = \{x_1x_2,x_1x_3,x_1x_4,x_2x_3,x_2x_4,x_2x_5\}$*. Then* $(\succ - P^-)$ *is shown in Figure 6-4(b) as the set of solid edges.* P^- *is not minimal because* $(P^- - \{x_1x_2\})$ *(Figure 6-4(c)) is also a full contractor of* $\succ$ *by CON. In fact,* $(P^- - \{x_1x_2\})$ *is a* minimal *full contractor of* $\succ$ *by CON. As we can see, having the edge* x_1x_2 *in* P^- *is not necessary. First, it is not a CON-edge. Second, the edge* x_2x_4 *of the CON-detour* $x_1 \succ x_2 \succ x_4$ *is already in* P^-*.*

As we have shown in Example 6.7, a minimal full contractor can be constructed by including in it only the edges which start some *CON*-detour, if the detour is not already disconnected. Thus, before adding such an edge to a full contractor, we need to know if an edge *not starting* that detour is already in the full contractor. Here we propose the following idea of computing a minimal full contractor. Instead of contracting $\succ$ by *CON* at once, split *CON* into *strata* , and contract $\succ$ incrementally by the strata of *CON*. A stratum of *CON* consists of only those edges whose detours can be disconnected simultaneously in a minimal way. The method of splitting a full contractor into strata we propose to use is as follows.

DEFINITION 6.5 *The* stratum index *of an edge $xy \in CON$ is* the maximum length of a $\succ$-path started by y and consisting of the end nodes of CON-edges. *A* stratum *is the set of all CON-edges with the same* stratum index .

This method of stratification has the following useful property. If a preference relation is contracted minimally by the strata with indices of up to n, then contracting that relation minimally by the stratum with the index $n+1$ minimally guarantees the minimality of the entire contraction.

Clearly, if a preference relation is infinite, a tuple can start $\succ$-paths of arbitrarily large lengths. Therefore, the stratum index of CON-edge may be undefined. We exclude such cases here, so we can assume that for each edge of CON relations, the stratum index is defined.

DEFINITION 6.6 *Let CON be a base contractor of a preference relation $\succ$. Let $K_{CON} = \{y \mid \exists x \ .\ xy \in CON\}$, and $\succ_{CON} = \succ \cap K_{CON} \times K_{CON}$. Then CON is* stratifiable *iff for every $y \in K_{CON}$ there is an integer k such that all the paths started by y in $\succ_{CON}$ are of length at most k. CON is* finitely stratifiable *iff there is a constant k such that all paths in $\succ_{CON}$ are of length at most k.*

Definition 6.6 implies that for every edge of stratifiable CON, the stratum index is defined. Since the shortest path in $\succ_{CON}$ is of length 0, the least stratum index for stratifiable relations is 0. Below we present an approach of constructing a minimal full contractor for a stratifiable relation CON.

THEOREM 6.2 **(Minimal full contractor construction).** *Let $\succ$ be a preference relation, and CON be a stratifiable base contractor of $\succ$. Let L_i be the set of the end nodes of all CON-edges of stratum i. Then P^-, defined as follows, is a* minimal full contractor *of $\succ$ by CON*

$$P^- = \bigcup_{i \in 0}^{\infty} E_i,$$

where

$$E_i = \{xy \mid \exists v \in L_i \;.\; xv \in CON \wedge x \succ y \wedge y \succeq v \wedge yv \notin (P^-_{i-1} \cup CON)\}$$

$$P^-_{-1} = \emptyset,$$

$$P^-_i = \bigcup_{j=0}^{i} E_i$$

Intuitively, the set E_i contains all the CON edges of stratum i along with the edges of $\succ$ which need to be discarded to contract the preference relation by that stratum. P^-_i is the union of all such sets up to stratum i.

PROOF OF THEOREM 6.2

Every E_i containts the CON-edges of stratum i. Thus, P^- contains CON. Now we prove that $(\succ - P^-)$ is an SPO. Its irreflexivity follows from the irreflexivity of $\succ$. Transitivity is proved by induction on stratum index.

It is given that $\succ$ is transitive. Now assume $(\succ - P^-_n)$ is transitive. Prove that $(\succ - P^-_{n+1}) = (\succ - P^-_n - E_{n+1})$ is transitive. For the sake of contradiction, assume

$$\exists x, y, z \;.\; xy \notin (\succ - P^-_{n+1}) \wedge xz, zy \in (\succ - P^-_{n+1}) \tag{1}$$

which implies

$$xz, zy \notin E_{n+1} \cup P^-_n \tag{2}$$

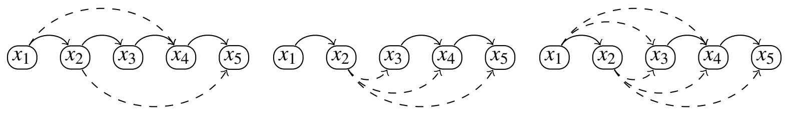
Transitivity of $(\succ - P^-_n)$ and (1) imply $xy \in (\succ - P^-_n)$ and thus $xy \in E_{n+1}$. Hence,

$$\exists v \in L_n \;.\; xv \in CON \wedge x \succ y \wedge y \succeq v \wedge yv \notin (P^-_n \cup CON) \tag{3}$$

According to (3), $y \succeq v$. If $y = v$, then (2) and (3) imply $xz \in E_{n+1}$ which is a contradiction.

If $y \succ v$, then $xz \notin E_{n+1}$ implies $zv \in P_n^- \cup CON$ by the construction of E_{n+1}. Note that $zv \in CON$ implies zv is a CON-edge of stratum index $n+1$ and thus either $zy \in E_{n+1}$ or $yv \in P_n^- \cup CON$, which contradicts (2) and (3). If $zv \in P_n^-$, then $zy, yv \notin P_n^-$ implies intransitivity of $(\succ - P_n^-)$, which contradicts the inductive assumption. Thus, P_{n+1}^- is a full contractor of $\succ$ by CON by induction. Now assume that $(\succ - P^-)$ is not transitive. Violation of transitivity means that there is an edge $xy \in P^-$ such that there exists a path from x to y none of whose edges is P^- (Lemma 6.1). Since xy must be in P_n^- for some n, that implies intransitivity of $(\succ - P_n^-)$, which is a contradiction. Thus P^- is a full contractor of $\succ$ by CON.

Now we prove that P^- is a *minimal* full contractor. If it is not, then by Theorem 6.1, there is $xy \in P^-$ for which there is no CON-detour which shares with P^- only the edge xy. Note that $xy \in P^-$ implies $xy \in E_n$ for some n. By definition of E_n, there is a CON-detour $x \succ y \succeq v$ which shares with P_n^- only xy. Since all CON-detours which xy belongs to have other P^--edges, $yv \in P^-$. Since $yv \notin P_n^-$, there must exist $k > n$ such that $yv \in E_k$. However, that is impossible by construction: every CON-detour which may be started by yv must have the stratum index not greater than n. □



(a) $\succ$ and CON (b) P_0^- (c) P_1^-

Figure 6-5: *Using Theorem 6.2 to compute a minimal full contractor*

EXAMPLE 6.8 *Let a preference relation $\succ$ be a total order of $\{x_1, \ldots, x_5\}$ (Figure 6-5(a), the transitive edges are omitted for clarity). Let a base contractor CON be $\{x_1x_4, x_2x_5\}$. We use Theorem 6.2 to construct a minimal full contractor of $\succ$ by CON. The relation CON has two strata: $L_0 = \{x_2x_5\}$, $L_1 = \{x_1x_4\}$. Then $E_0 = \{x_2x_3, x_2x_4, x_2x_5\}$, $P_0^- = E_0$, $E_1 = \{x_1x_3, x_1x_4\}$, $P_1^- = E_0 \cup E_1$, and a minimal full contractor of $\succ$ by CON is $P^- = P_1^-$.*

It is easy to observe that the full contractor P^- constructed in Theorem 6.2 has the property that its every edge *starts* at least one CON-detour in which xy is the only P^--edge. Full contractors which have this property are called *prefix*. Prefix full contractors are minimal by Theorem 6.1. It turns out that a prefix full contractor is unique for a given preference relation and a given base contractor.

PROPOSITION 6.1 *Given a preference relation $\succ$ and a base contractor CON stratifiable, there exists a unique prefix full contractor P^- of $\succ$ by CON.*

PROOF

The existence of a prefix full contractor follows from Theorem 6.2. The fact that every prefix full contractor is equal to P^- constructed by Theorem 6.2 can be proved by induction in CON stratum index. Namely, we show that for every n, P_n^- is contained in any prefix full contractor of $\succ$ by CON. Clearly, the set E_0 contracting $\succ$ by the 0^{th} stratum of CON has to be in any prefix full contractor. Assume every edge in P_n^- is in any prefix full contractor of $\succ$ by CON. If an edge $xy \in E_{n+1} - CON$, then there is a CON-detour $x \succ y \succ v$ in which xy is the only P^--edge (i.e., $yv \notin P^-$). Hence if xy is not in some prefix full contractor P', then yv has to be in P' by Lemma 6.1. However, $P_n^- \subset P'$ is enough to disconnect every CON-detour with the stratum index up to n, and yv can only start a CON-detour with the stratum index up to n. Hence P' is not a minimal full contractor and P^- is a unique prefix full contractor. $\square$

6.6 Contraction by finitely stratifiable relations

In this section, we consider practical issues of computing minimal full contractors. In particular, we show how the method of constructing a prefix full contractor we have proposed in Theorem 6.2 can be adopted to various classes of preference and base contractor relations. Note that the definition of the minimal full contractor in Theorem 6.2 is recursive.

Namely, to find the edges we need to discard for contracting the preference relation by the stratum $n+1$ of *CON*, we need to know which edges to discard for contracting it by all the previous strata. It means that for base contractor relations which are not finitely stratifiable (i.e., *CON* has infinite number of strata), the corresponding computation will never terminate.

Now assume that *CON* is a finitely stratifiable relation. First we note that any base contractor of a finite preference relation is finitely stratifiable: all paths in such preference relations are not longer than the size of the relation, and base contractors are required to be subsets of the preference relations. At the same time, if *CON* is a base contractor of an infinite preference relation, then the finite stratification property of *CON* does not imply the finiteness of *CON*. In particular, it may be the case that the *length* of all paths in $\succ_{CON}$ is bounded, but the *number* of paths is infinite. This fact is illustrated in the next example.

EXAMPLE 6.9 *Let a preference relation $\succ$ be defined as $o \succ o' \equiv o.price < o'.price$. Let every tuple have two Q-attributes: price and year. Let also the base contractor relations CON_1 and CON_2 be defined as*

$$CON_1(o,o') \equiv o.price < 1 \wedge (o'.price = 2 \vee o'.price = 3),$$
$$CON_2(o,o') \equiv o.price < 1 \wedge o'.price \geq 2.$$

then

$$K_{CON_1} \equiv \{o \mid o.price = 2 \vee o.price = 3\}$$
$$K_{CON_2} \equiv \{o \mid o.price \geq 2\}$$

and

$$o \succ_{CON_1} o' \equiv o.price = 2 \wedge o'.price = 3$$
$$o \succ_{CON_2} o' \equiv o.price \geq 2 \wedge o'.price > 2 \wedge o.price < o'.price$$

Then CON_1 is finitely stratifiable *since despite the fact that the number of edges in $\succ_{CON_1}$ and in CON_1 is infinite (due to the infiniteness of the domain of year), the length of the longest path in $\succ_{CON_1}$ equals to* 1*. Such paths are started by tuples with the value of price equal to* 2 *and ended by tuples with price equal to* 3*. At the same time, CON_2 is not finitely stratifiable since price is a Q-attribute and thus there is no constant bounding the length of all paths in $\succ_{CON_2}$.*

Below we consider the cases of finite and finitely representable finitely stratifiable base contractors separately.

6.6.1 Computing prefix full contractor: finitely representable relations

Here we assume that the relations CON and $\succ$ are represented by finite ERO-formulas F_{CON} and $F_{\succ}$. We aim to construct a finite ERO-formula F_{P^-} which represents a prefix full contractor of $\succ$ by CON. The function `minContr(`$F_{\succ}$`,` F_{CON}`)` shown below exploits the method of constructing prefix full contractors from Theorem 6.2 adopted to formula representations of relations. All the intermediate variables used in the algorithm store formulas. Hence, for example, any expression in the form $''F(x,y) := \ldots''$ means that the formula-variable F is assigned the formula written in the right-hand side, which has two free tuple variables x and y. The operator QE used in the algorithm computes a quantifier-free formula equivalent to its argument formula. For ERO-formulas, the operator QE runs in time polynomial in the size of its argument formula (if the number of attributes in $\mathcal{A}$ is fixed), and exponential in the number of attributes in $\mathcal{A}$.

To compute formulas representing different strata of *CON*, the function `getStratum` is used. It takes three parameters: the formula $F_{\succ_{CON}}$ representing the relation $\succ_{CON}$, the formula $F_{K_{CON}}$ representing the set of the end nodes of *CON*-edges, and the stratum index i. It returns a formula which represents the set of the end nodes of *CON*-edges of stratum i , or `undefined` if the corresponding set is empty. That formula is computed according to the definition of a stratum.

PROPOSITION 6.2 *Let CON be a finitely stratifiable base contractor of a preference relation $\succ$. Then Algorithm 6.1 terminates and computes a prefix full contractor of $\succ$ by CON.*

Algorithm 6.1 `minContr`($F_{\succ}$, F_{CON})

```
 1: $i = 0$
 2: $F_{P^-_{-1}}(x,y) := false$
 3: $F_{K_{CON}}(y) := QE(\exists x \,.\, F_{CON}(x,y))$
 4: $F_{\succ_{CON}}(x,y) := F_{CON}(x,y) \wedge F_{K_{CON}}(x) \wedge F_{K_{CON}}(y)$
 5: $F_{L_i}(y) :=$ getStratum($F_{\succ_{CON}}, F_{K_{CON}}, i$)
 6: while $F_{L_i}$ is defined do
 7:     $F_{E_i}(x,y) := QE(\exists v \,.\, F_{L_i}(v) \wedge F_{CON}(x,v) \wedge F_{\succ}(x,y) \wedge$
                      $(y = v \vee F_{\succ}(y,v) \wedge \neg(F_{P^-_{i-1}}(y,v) \vee F_{CON}(y,v))))$
 8:     $F_{P^-_i}(x,y) := F_{P^-_{i-1}}(x,y) \vee F_{E_i}(x,y)$
 9:     $i := i + 1;$
10:     $F_{L_i}(y) :=$ getStratum($F_{\succ_{CON}}, F_{K_{CON}}, i$)
11: end while
12: return $P^-_i$
```

Algorithm 6.2 `getStratum`($F_{\succ_{CON}}$, $F_{K_{CON}}$, i)

```
Require: $i \geq 0$
 1: if i = 0 then
 2:    $F_{L_i}(y) := QE(\ F_{K_{CON}}(y) \wedge \neg\exists x_1 (F_{\succ_{CON}}(y, x_1)))$
 3: else
 4:    $F_{L_i}(y) := QE(\ \exists x_1, \ldots, x_i\ .\ F_{\succ_{CON}}(y, x_1) \wedge F_{\succ_{CON}}(x_1, x_2) \wedge \ldots \wedge F_{\succ_{CON}}(x_{i-1}, x_i)) \wedge$
                   $\neg\exists x_1, \ldots, x_{i+1}\ .\ F_{\succ_{CON}}(y, x_1) \wedge F_{\succ_{CON}}(x_1, x_2) \wedge \ldots \wedge F_{\succ_{CON}}(x_i, x_{i+1})))$
 5: end if
 6: if $\exists y\ .\ F_{L_i}(y)$ then
 7:    return $F_{L_i}$
 8: else
 9:    return undefined
10: end if
```

Proposition 6.2 holds because Algorithm 6.1 uses the construction from Theorem 6.2. Below we show an example of computing a prefix full contractor for a finitely representable preference relation.

EXAMPLE 6.10 *Let a preference relation $\succ$ be defined by the following formula*

$$F_{\succ}(o, o') \equiv o.m = BMW \wedge o'.m = VW \vee o.m = o'.m \wedge o.price < o'.price$$

and a base contractor CON be defined by

$$F_{CON}(o, o') \equiv o.m = o'.m \wedge ((11000 \leq o.price \leq 13000 \wedge o'.price = 15000) \vee \\ (10000 \leq o.price \leq 12000 \wedge o'.price = 14000))$$

where m is a $\mathcal{C}$-attribute and price is a $\mathcal{Q}$-attribute. Then $F_{K_{CON}}(o) \equiv o.price = 14000 \vee$

$o.price = 15000$ *and* $F_{\succ_{CON}}(o,o') \equiv F_{\succ}(o,o') \wedge F_{K_{CON}}(o) \wedge F_{K_{CON}}(o')$. *The end nodes of the CON strata are defined by the following formulas:*

$$F_{L_0}(o) \equiv o.price = 15000 \wedge o.m \neq BMW$$
$$F_{L_1}(o) \equiv o.price = 15000 \wedge o.m = BMW \vee o.price = 14000 \wedge o.m \neq BMW$$
$$F_{L_2}(o) \equiv o.price = 14000 \wedge o.m = BMW.$$

The relations contracting all CON strata are defined by the following formulas

$$\begin{aligned} F_{E_0}(o,o') \equiv\, & o.m = o'.m \neq BMW \wedge 11000 \leq o.price \leq 13000 \wedge 13000 < o'.price \leq 15000 \\ F_{E_1}(o,o') \equiv\, & o.m = o'.m = BMW \wedge 11000 \leq o.price \leq 13000 \wedge 13000 < o'.price \leq 15000 \vee \\ & o.m = o'.m \neq BMW \wedge 10000 \leq o.price < 11000 \wedge 13000 < o'.price \leq 14000 \\ F_{E_2}(o,o') \equiv\, & o.m = o'.m = BMW \wedge 10000 \leq o.price \leq 11000 \wedge 13000 < o'.price \leq 14000 \end{aligned}$$

Finally, a full contractor P^- of $\succ$ by CON is defined by

$$\begin{aligned} F_{P^-}(o,o') \equiv o.m = o'.m \wedge (& 11000 \leq o.price \leq 13000 \wedge 13000 < o'.price \leq 15000 \vee \\ & 10000 \leq o.price < 11000 \wedge 13000 < o'.price \leq 14000) \end{aligned}$$

We note that the finite stratification property of *CON* is crucial for the termination of the algorithm: the algorithm does not terminate for not finitely stratifiable relations.

6.6.2 Computing prefix full contractor: finite relations

In this section, we consider finite relations $\succ$ and *CON*. We assume that the relations are stored in separate tables: a preference relation table R and a base contractor table C, each having two columns X and Y. Every tuple in a table corresponds to an element of the corresponding binary relation. Hence, R has to be an SPO and $C \subseteq R$. Here we present an algorithm of computing a prefix full contractor of a preference relation $\succ$ by CON represented by such tables. Essentially, the algorithm is an adaptation of Theorem 6.2.

Algorithm 6.3 `minContrFinite`(R, C)

```
Require: $R$ is transitive, $C \subseteq R$
 1: $P \leftarrow C$
 2: /* Get the end nodes of all $C$-edges */
 3: $EC \leftarrow \pi_Y(C)$
 4: /* $RC$ is related to $R$ as $\succ_{CON}$ to $\succ$ in Definition 6.6 */
 5: $RC \leftarrow \pi_{R.X,\,R.Y}\ (EC_1 \underset{EC_1.Y=R.X}{\bowtie} R \underset{EC_2.Y=R.Y}{\bowtie} EC_2)$
 6: while $EC$ not empty do
 7:     /* Get the end nodes of the next stratum $C$-edges */
 8:     $E \leftarrow EC - \pi_X(RC)$
 9:     /* Prepare $EC$ and $RC$ for the next iteration */
10:     $EC \leftarrow EC - E$
11:     $RC \leftarrow RC - RC \underset{RC.Y=E.Y}{\bowtie} E$
12:     /* Add to $P$ the $R$-edges contracting the current stratum of $C$*/
13:     $P \leftarrow P \cup \pi_{R_1.X,\,R_1.Y}\ (R_1 \underset{R_1.Y\,=\,R_2.X}{\bowtie} (R_2 - P) \underset{R_1.X\,=\,C.X,\,R_2.Y\,=\,C.Y}{\bowtie} (C \underset{C.Y\,=\,E.Y}{\bowtie} E))$
14: end while
15: return $P$
```

The function `minContrFinite` takes two arguments: a table R and a table C. The function is implemented in terms of relational algebra operators. First, it constructs two tables: EC storing the end nodes of all C-edges, and RC storing a restriction of the original preference relation R to EC. These two tables are needed for obtaining the strata of C. After that, the function picks all strata of C one by one and contracts the original preference relation by each stratum in turn, as shown in Theorem 6.2.

The extraction of the strata of *CON* in the order of the stratum index is performed as follows. It is clear that the nodes ending *CON*-edges of stratum 0 do not start any edge in *RC*. The set *E* computed in line 8 is a difference of the set *EC* of the nodes ending *C*-edges and the nodes starting some edges in *RC*. Hence, *E* stores all the nodes ending *C*-edges of stratum 0. To get the end nodes of the next stratum of *C*, we need remove all the edges from *RC* which end in members of *E*, and remove *E* from *EC*. After the stratum with the highest index is obtained, the relation *EC* becomes empty.

PROPOSITION 6.3 *Algorithm 6.3 computes a prefix full contractor of R by C. Its running time is* $O(|C|^2 \cdot |R| \cdot log|R|)$.

Proposition 6.3 holds because Algorithm 6.3 uses the construction from Theorem 6.2. The stated running time may be obtained by applying some simple optimizations: (i) sorting *EC* after constructing it (line 3), (ii) sorting on X, Y the table *R* and the table *RC* right after its construction (line 5), (iii) keeping these relations sorted after every change. In addition to that, we store the relation *P* containing the intermediate full contractor edges as a copy of *R*, in which the edges which belong to the prefix full contractor are marked. By doing so, *P* is maintained in the sorted state throughout the algorithm.

6.7 Preference-protecting contraction

Consider the operation of minimal preference contraction described above. In order to contract a preference relation, a user has to specify a base contractor *CON*. The main criteria we use to define a contracted preference relation is *minimality of preference change*. However, a minimal full contractor P^- may contain additional preferences which are not in *CON*. So far, we have not paid attention to the contents of P^-, assuming that any minimal full contractor is equally good for a user. However, this may not be the case in real life. Assume that an original preference relation $\succ$ is combined from two preference relations

$\succ = \succ_{old} \cup \succ_{recent}$, where $\succ_{old}$ describes user preferences introduced by the user a long time ago, and $\succ_{recent}$ describes more recent preferences. Now assume that the user wants to contract $\succ$ by CON, at least two minimal full contractors are possible: P_1^- which consists of CON and some preferences of $\succ_{old}$, and P_2^- consisting of CON and some preferences of $\succ_{recent}$. Since $\succ_{recent}$ has been introduced recently, discarding members of $\succ_{old}$ may be more reasonable then members of $\succ_{recent}$. Hence, sometimes there is a need to compute full contractors which *protect* some existing preferences from removal.

Here we propose an operator of *preference-protecting* contraction. In addition to a base contractor CON, a subset P^+ of the original preference relation to be *protected from removal* in the contracted preference relation may also be specified. Such a relation is complementary with respect to the base contractor: the relation CON defines the preferences to discard, whereas the relation P^+ defines the preferences to protect.

DEFINITION 6.7 *Let $\succ$ be a preference relation and CON be a base contractor of $\succ$. Let a relation P^+ be such that $P^+ \subseteq \succ$. A full contractor P^- of $\succ$ by CON such that $P^+ \cap P^- = \emptyset$ is called a P^+-*protecting full contractor of $\succ$ by CON. *A minimal full contractor P^- of $\succ$ by CON such that $P^+ \cap P^- = \emptyset$ is called a P^+-*protecting minimal full contractor of $\succ$ by CON.

Given any full contractor P^- of $\succ$ by CON, by Lemma 6.1, P^- must contain at least one edge from every CON-detour. Thus, if P^+ contains an entire CON-detour, protecting P^+ while contracting $\succ$ by CON is not possible.

THEOREM 6.3 *Let CON be a stratifiable base contractor relation of a preference relation $\succ$ such that $P^+ \subset \succ$. There exists a minimal full contractor of $\succ$ by CON that protects P^+ if and only if $TC(P^+) \cap CON = \emptyset$.*

As we noted, the necessary condition of the theorem above follows from Lemma 6.1. The sufficient condition follows from Theorem 6.4 we prove further.

A naive way of computing a preference-protecting minimal full contractor is by finding a minimal full contractor P^- of $(\succ - P^+)$ and then adding P^+ to P^-. However, $(\succ - P^+)$ is not an SPO in general, thus obtaining SPO of $\succ - (P^- \cup P^+)$ becomes problematic.

The solution we propose here uses the following idea. First, we find a base contractor CON' such that minimal contraction of $\succ$ by CON' is equivalent to minimal contraction of $\succ$ by CON with protected P^+. After that, we compute a minimal full contractor of $\succ$ by CON' using Theorem 6.2.

Recall that minimal full contractors constructed in Theorem 6.2 are *prefix*, i.e., every edge xy in such a full contractor starts some CON-detour in which xy is the only edge of the contractor. Thus, if no member of P^+ starts a CON-detour in $\succ$, then the minimal full contractor and P^+ have no common edges. Otherwise assume that an edge $xy \in P^+$ starts a CON-detour in $\succ$. By Lemma 6.1, any P^+-protecting full contractor P^- has to contain an edge different from xy which belongs to CON-detours started by xy. Moreover, for CON-detours of length two started by xy, P^- has to contain the *edges ending those CON-detours*. Such a set of edges is defined as follows:

$$Q = \{xy \mid \exists u : u \succ x \succ y \wedge uy \in CON \wedge ux \in P^+\}.$$

It turns out that the set Q is not only contained in any P^+-protecting full contractor, but it can also be used to construct a P^+-protecting minimal full contractor as shown in the next theorem.

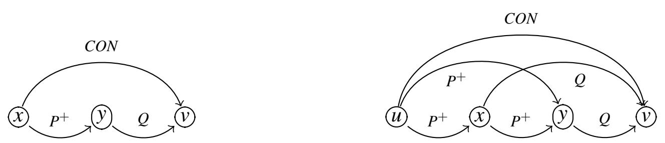
THEOREM 6.4 *Let $\succ$ be a preference relation, and CON be a stratifiable base contractor of $\succ$. Let also P^+ be a transitive relation such that $P^+ \subseteq \succ$ and $P^+ \cap CON = \emptyset$. Then the prefix full contractor of $\succ$ by $CON \cup Q$ is a P^+-protecting minimal full contractor of $\succ$ by CON.*

PROOF

Let P^- be a prefix full contractor of $\succ$ by $CON' = CON \cup Q$. We prove that $P^- \cap P^+ = \emptyset$, i.e., P^- protects P^+. For the sake of contradiction, assume there is $xy \in P^+ \cap P^-$. We show that this contradicts the prefix property of P^-. Since P^- is a prefix full contractor, there is a CON'-detour from x to some v in $\succ$, started by xy and having only the edge xy in P^-. We have two choices: either it is a CON-detour or a Q-detour. Consider the first case. Clearly, $y \neq v$, otherwise $P^+ \cap CON \neq \emptyset$. Thus, $xv \in CON$ and $x \succ y \succ v$ (Figure 6-6(a)). $yv \in Q$ follows from $xy \in P^+$, $xv \in CON$ and the construction of Q. Note that every path from y to v in $\succ$ contains a P^--edge because P^- is a full contractor of $\succ$ by $CON \cup Q$. That implies that no CON-detour from x to v started by xy has only xy in P^- which contradicts the initial assumption.

Consider the second case, i.e., there is a Q-detour from x to some v started by xy and having only the edge xy in P^-. Since $xv \in Q$, there is $uv \in CON$ such that $ux \in P^+$ (Figure 6-6(b)). $ux, xy \in P^+$ imply $uy \in P^+$ by transitivity of P^+. $uy \in P^+$ and $uv \in CON$ imply $yv \in Q$. That along with the fact that P^- is a full contractor of $\succ$ by $CON \cup Q$ implies that every path in $\succ$ from y to v contains a P^--edge. Hence, there is no Q-detour from x to v started by xy and having only xy in P^-. That contradicts the initial assumption about xy.

Now we prove that P^- is a minimal full contractor of $\succ$ by CON. The fact that it is a full contractor of $\succ$ by CON follows from the fact that it is a full contractor of $\succ$ by a superset CON' of CON. We prove now its minimality. Since P^- is a prefix full contractor of $\succ$ by CON', for every $xy \in P^-$, there is $xv \in CON'$ such that there is a corresponding detour T in which xy is the only P^--edge. If it is a CON-detour, then xy satisfies the minimality condition from Theorem 6.1. If it is a Q-detour, then there is a CON-edge uv such that $ux \in P^+$. We showed above that P^- protects P^+. Hence, the CON-detour obtained by joining the edge ux and T has only xy in P^-. Therefore, P^- is a minimal full contractor of $\succ$ by CON. $\square$



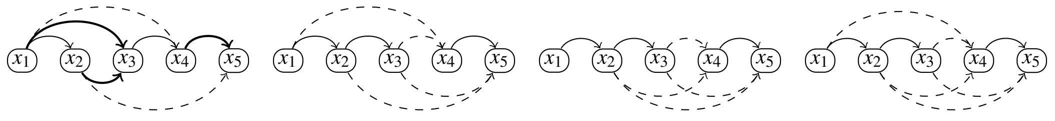
(a) P^+-protection: case 1 (b) P^+-protection: case 2

Figure 6-6: *Proof of Theorem 6.4*

Note that the sets of the end nodes of $(CON \cup Q)$-edges and the end nodes of CON-edges coincide by the construction of Q. Therefore, $(CON \cup Q)$ is stratifiable or finitely stratifiable if and only if CON is stratifiable or finitely stratifiable, correspondingly. Hence, if CON is a finitely stratifiable relation with respect to $\succ$, Algorithms 6.1 and 6.3 can be used to compute a preference-protecting minimal full contractor of $\succ$ by CON. If the relations $\succ$ and CON are finite, then Q can be constructed in polynomial time in the size of $\succ$ and CON by a relational algebra expression constructed from its definition. If the relations are finitely representable, then Q may be computed using the quantifier elimination operator QE.

For Theorem 6.4 to apply, the relation P^+ has to be transitive. Non-transitivity of P^+ implies that there are two edges $xy, yz \in P^+$ which should be protected while transitive edge xz is not critical. However, a relation obtained as a result of preference-protecting contraction is a preference relation (i.e., SPO). Hence, the edge xz will also be protected in the resulting preference relation. This fact implies that protecting any relation is equivalent to protecting its minimal transitive extension: its transitive closure. Therefore, if P^+ is not transitive, one needs to compute its transitive closure to use Theorem 6.4. For finite relations, transitive closure can be computed in polynomial time [CLRS01]. For finitely representable relations, *Constraint Datalog* [KKR95] can be used to compute transitive closure.

Another important observation here is that the P^+-protecting minimal full contractor of $\succ$ by CON computed according to Theorem 6.4 is not necessary a *prefix* full contractor of



(a) $\succ$, *CON*, and P^+ (b) $\succ$ and CON' (c) P_0^- (d) P_1^-

Figure 6-7: *Using Theorem 6.4 to compute a preference-protecting minimal full contractor*

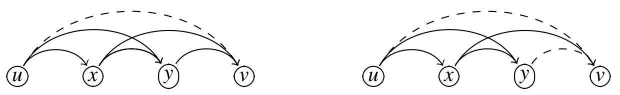
$\succ$ by *CON*. This fact is illustrated in the following example.

EXAMPLE 6.11 *Let a preference relation $\succ$ be a total order of $\{x_1,\ldots,x_5\}$ (Figure 6-7(a), the transitive edges are omitted for clarity). Let a base contractor CON be $\{x_1x_4, x_2x_5\}$, and $P^+ = \{x_1x_3, x_2x_3, x_4x_5\}$.*

The existence of a minimal P^+-protecting full contractor of $\succ$ by CON follows from Theorem 6.3. We use Theorem 6.4 to construct it. The set Q is equal to $\{x_3x_4, x_3x_5\}$ and $CON' = \{x_1x_4, x_2x_5, x_3x_4, x_3x_5\}$. We construct a prefix full contractor of $\succ$ by CON'. The relation CON' has two strata: $L_0 = \{x_2x_5, x_3x_5\}$, $L_1 = \{x_1x_4, x_3x_4\}$. Then $E_0 = \{x_2x_5, x_3x_5, x_2x_4, x_3x_4\}$, $P_0^- = E_0$, $E_1 = \{x_1x_4, x_3x_4\}$, $P_1^- = E_0 \cup E_1$, and $P^- = P_1^-$. By Theorem 6.4, P^- is a P^+-protecting minimal full contractor of $\succ$ by CON. However, P^- is not a prefix full contractor of $\succ$ by CON, because the edges x_3x_4, x_3x_5 do not start any CON-detour.

6.8 Meet preference contraction

In this section, we consider the operation of *meet preference contraction*. In contrast to the preceding sections, where the main focus was the minimality of preference relation change, the contraction operation considered here changes a preference relation not necessarily in a minimal way. A full meet contractor of a preference relation is semantically a *union* of all minimal sets of reasons of discarding a given set preferences. When a certain set of preferences is required to be protected while contracting a preference relation, the operation



(a) $\succ$ and CON_1 (b) $\succ$ and CON_2

Figure 6-8: *Example 6.12*

of *preference-protecting meet contraction* may be used.

DEFINITION 6.8 *Let $\succ$ be a preference relation, CON a base contractor of $\succ$, and $P^+ \subseteq \succ$. The relation P^m is a* full meet contractor *of $\succ$ by CON iff*

$$P^m = \bigcup_{P^- \in \mathcal{P}^m} P^-,$$

for the set $\mathcal{P}^m$ of all minimal full contractors of $\succ$ of CON. The relation P^m_{P+} is a full P^+-protecting meet contractor of $\succ$ by CON *iff*

$$P^m_{P+} = \bigcup_{P^- \in \mathcal{P}^m_{P+}} P^-,$$

for the set $\mathcal{P}^m_{P+}$ of all P^+-protecting minimal full contractors of $\succ$ of CON.

Note that the relations $(\succ - P^m)$ and $(\succ - P^m_{P+})$ can be represented as intersections of preference (i.e., SPO) relations and thus are also preference (i.e., SPO) relations. Let us first consider the problem of constructing full meet contractors.

By the definition above, an edge xy is in the full meet contractor of a preference relation $\succ$ by CON if there is a minimal full contractor of $\succ$ by CON which contains xy. Theorem 6.1 implies that if there is no CON-detour in $\succ$ containing xy, then xy is not in the corresponding full meet contractor. However, the fact that xy belongs to a CON-detour is not a sufficient condition for xy to be in the corresponding full meet contractor.

EXAMPLE 6.12 *Let a preference relation $\succ$ be a total order of $\{u,x,y,v\}$. Let also $CON_1 = \{uv\}$ (Figure 6-8(a)) and $CON_2 = \{uv, yv\}$ (Figure 6-8(b)). There is only one*

CON_1- and CON_2-detour containing xy: $u \succ x \succ y \succ v$. There is also a minimal full contractor of $\succ$ by CON_1 which contains xy: $P_1^- = \{uy, xv, xy, uv\}$. However, there is no minimal full contractor of $\succ$ by CON_2 which contains xy because the edge yv of the CON_2-detour $u \succ x \succ y \succ v$ is in CON_2.

In Theorem 6.5, we show how full meet contractors can be constructed in the case of *finitely stratifiable base contractors* . According to that theorem, a $\succ$-edge xy is in the full meet contractor of $\succ$ by CON if and only if there is a full contractor P^- of $\succ$ by CON such that xy is the only P^--edge in some CON-detour. We use Theorem 6.3 to show that there is a minimal full contractor of $\succ$ by CON which contains xy while the other edges of the detour are protected.

THEOREM 6.5 *Let CON be a finitely stratifiable base contractor of a preference relation $\succ$. Then the full meet contractor of $\succ$ by CON is*

$$P^m = \{xy \mid \exists uv \in CON \,.\, u \succeq x \succ y \succeq v \wedge$$
$$(ux \in (\succ - CON) \vee u = x) \wedge (yv \in (\succ - CON) \vee y = v)\}$$

PROOF

By Corollary 6.1, an edge xy is in a minimal full contractor P^- of $\succ$ by CON, if there is CON-detour of at most three edges in $\succ$ in which xy is the only P^--edge. Hence any minimal full contractor is a subset of P^m. Now take every edge xy of P^m and show there is a minimal full contractor of $\succ$ by CON which contains xy. Let $u \succeq x \succ y \succeq v$ for $uv \in CON$. Let us construct a set P' as follows:

$$P' = \begin{cases} \{ux, yv\} & \text{if } u \succ x \wedge y \succ v \\ \{ux\} & \text{if } u \succ x \wedge y = v \\ \{yv\} & \text{if } u = x \wedge y \succ v \\ \emptyset & \text{if } u = x \wedge y = v \end{cases}$$

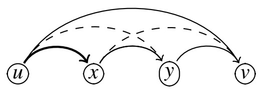


Figure 6-9: *$\succ$, CON, and P^+ from Example 6.13*

P' is transitive, $P' \cap CON = \emptyset$, and $P' \subseteq \succ$. Theorem 6.3 implies that there is a P'-protecting minimal full contractor P^- of $\succ$ by CON. Since P^- protects P', there is a CON-detour in $\succ$ from u to v in which xy is the only P^--edge. This implies that $xy \in P^-$. □

Now consider the case of P^+-protecting full meet contractors. A naive solution is to construct it as the difference of P^m defined above and P^+. However, in the next example we show that such solution does not work in general.

EXAMPLE 6.13 *Let a preference relation $\succ$ be a total order of $\{u,x,y,v\}$ (Figure 6-9). Let also $CON = \{uy,xv\}$ and $P^+ = \{ux\}$. Note that $yv \notin P^+$, and by Theorem 6.5, $yv \in P^m$. Hence, $yv \in (P^m - P^+)$. However, note that $ux \in P^+$ implies that xy must be a member of every P^+-protecting full contractor in order to disconnect the path from u to y. Hence, there is no CON-detour in which yv is the only edge of the full contractor, and yv is not a member of any P^+-protecting full contractor.*

The next theorem shows how a P^+-protecting full contractor may be constructed. The idea is similar to Theorem 6.5. However, to construct a full meet contractor, we used the set CON as a common part of all minimal full contractors. In the case of P^+-protecting full meet contractor, a superset C_{P^+} of CON is contained in all of them. Such a set C_{P^+} may be viewed as a union of CON and the set of all edges of $\succ$ that *must be discarded due to the protection of P^+*.

THEOREM 6.6 *Let CON be a finitely stratifiable base contractor of a preference relation $\succ$, and P^+ a transitive relation such that $P^+ \subseteq \succ$ and $P^+ \cap CON = \emptyset$. Then the*

P^+-protecting full meet contractor of $\succ$ by CON is

$$P^m_{P^+} = \{xy \mid xy \notin P^+ \land \exists uv \in CON \;.\; u \succeq x \succ y \succeq v \land$$
$$(ux \in (\succ - C_{P^+}) \lor u = x) \land (yv \in (\succ - C_{P^+}) \lor y = v)\},$$

for

$$C_{P^+} = \{xy \mid \exists uv \in CON \;.\; u \succeq x \succ y \succeq v \land (ux \in P^+ \lor u = x)$$
$$\land (yv \in P^+ \lor y = v)\}$$

PROOF

First, it is easy to observe that C_{P+} is a subset of any P^+-protecting full contractor of $\succ$ by *CON*. It is constructed from the edges *xy* which participate in *CON*-detours of length at most three where all the other edges have to be protected. Since every *CON*-detour has to have at least one edge in a full contractor, *xy* has to be a member of every full contractor.

We show that every P^+-protecting minimal full contractor P^- of $\succ$ by *CON* is a subset of $P^m_{P^+}$. If some $xy \in P^-$, then by Corollary 6.1 there is an edge $uv \in CON$ such that $u \succeq x \succ y \succeq v$ and $ux, yv \notin P^-$. We show that $xy \in P^m_{P^+}$. That holds if $xy \notin P^+$ (which holds for P^- by definition) and

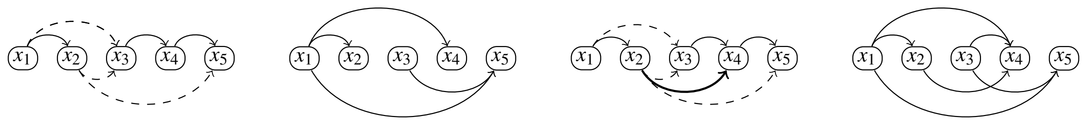
$$(ux \in (\succ - C_{P^+}) \lor u = x) \land (yv \in (\succ - C_{P^+}) \lor y = v)$$

If both $u = x$ and $y = v$ hold then the expression above holds. Now assume $u \succ x$ (the case $y \succ v$ is similar). If $ux \in C_{P^+}$ then, as we showed above, $ux \in P^-$ which is a contradiction. Hence, $ux \in (\succ - C_{P^+})$ and $xy \in P^m_{P^+}$. Finally, $P^- \subseteq P^m_{P^+}$.

Now we show that every $xy \in P^m_{P^+}$ is contained in every P^+-protecting minimal full contractor of $\succ$ by *CON*. The proof is similar to the proof of Theorem 6.5. By definition of

$P^m_{P^+}$, take xy such that $u \succeq x \succ y \succeq v$. Construct the set P' for xy as in the proof of Theorem 6.5. We show that for the set $P'' = TC(P^+ \cup P')$ we have $P'' \cap CON = \emptyset$. For the sake of contradiction, assume $P'' \cap CON \neq \emptyset$. This implies that there is a CON-detour consisting of P^+ and P' edges. Having only P^+-edges in the detour contradicts the initial assumption that $P^+ \cap CON = \emptyset$. Having a single edge of P' in the detour implies that the edge (either ux or yv) is in C_{P^+}, which contradicts the definition of $P^m_{P^+}$. Having both ux and yv in the detour implies that $xy \in P^+$ which also contradicts the definition of $P^m_{P^+}$. Hence, $P'' \cap CON = \emptyset$, and by Theorem 6.4, there is a P''-protecting minimal full contractor P^- of $\succ$ by CON which is also a P^+-protecting minimal full contractor. Since there is a CON-detour in which xy is unprotected by P^-, $xy \in P^-$. $\square$

We note that given the expressions for the meet and P^+-protecting full meet contractors in Theorems 6.5 and 6.6, one can easily obtain such contractors for finite and finitely representable relations: by evaluation of a relational algebra query in the former case and by quantifier elimination in the latter case.



(a) $\succ$ and CON (b) $\succ - P^m$ (c) $\succ$, CON, and P^+ (d) $\succ - P^m_{P^+}$

Figure 6-10: *Computing full meet contractor and P^+-protecting full meet contractor*

EXAMPLE 6.14 *Let a preference relation $\succ$ be a total order of $\{x_1, \dots, x_5\}$ (Figure 6-10(a), the transitive edges are omitted for clarity). Let a base contractor CON be $\{x_1x_3, x_2x_3, x_2x_5\}$, and $P^+ = \{x_2x_4\}$.*

A full meet contractor P^m of $\succ$ by CON is $\{x_1x_3, x_2x_3, x_2x_5, x_2x_4, x_3x_4, x_4x_5\}$. The resulting contracted preference relation is shown on Figure 6-10(b). A P^+-protecting full meet contractor of $\succ$ by CON is $\{x_1x_3, x_2x_3, x_2x_5, x_4x_5\}$. The resulting contracted preference relation is shown on Figure 6-10(d). Note that C_{P^+} here is $CON \cup \{x_4x_5\}$.

6.9 Querying with contracted preferences

In this section, we show some new techniques which can be used to optimize the evaluation of the winnow operator under contracted preferences.

In user-guided preference modification frameworks [Cho07b, BGS06], it is assumed that users alter their preferences after examining sets of the most preferred tuples returned by winnow. Thus, if preference contraction is incorporated into such frameworks, there is a need to compute winnow under contracted preference relations. Here we show how the evaluation of winnow can be optimized in such cases.

Let $\succ$ be a preference relation, CON be a base contractor of $\succ$, P^- be a full contractor of $\succ$ by CON, and the contracted preference relation $\succ' = (\succ - P^-)$. Denote the set of the starting and the ending tuples of R-edges for a binary relation R as $S(R)$ and $E(R)$ correspondingly.

$$S(R) = \{x \mid \exists y \,.\, xy \in R\}$$
$$E(R) = \{y \mid \exists x \,.\, xy \in R\}$$

Let us also define the set $M(CON)$ of the tuples which participate in CON-detours in $\succ$

$$M(CON) = \{y \mid \exists x, z \,.\, x \succ y \wedge xz \in CON \wedge y \succeq z\}$$

Assume we also know quantifier-free formulas $F_{S(P^-)}$, $F_{E(P^-)}$, $F_{M(CON)}$, and $F_{S(CON)}$ representing these sets for P^- and CON. Then the following holds.

PROPOSITION 6.4 *Given a finite set of tuples r*

1. $w_{\succ}(r) \subseteq w_{\succ'}(r)$

2. *If $\sigma_{F_{S(P^-)}}(w_{\succ}(r)) = \emptyset$, then $w_{\succ}(r) = w_{\succ'}(r)$.*

3. *If P^- is a prefix full contractor, then* $\sigma_{F_{S(P^-)}}(r) = \sigma_{F_{S(CON)}}(r)$

4. $w_{\succ'}(r) = w_{\succ'}(w_{\succ}(r) \cup \sigma_{F_{E(P^-)}}(r))$

`PROOF`

1. By definition, $w_{\succ}(r)$ contains the set of undominated tuples w.r.t the preference relation $\succ$. Thus, $\succ' \subset \succ$ implies that if a tuple o was undominated w.r.t $\succ$, it will be undominated w.r.t $\succ'$, too. Hence, $w_{\succ}(r) \subseteq w_{\succ'}(r)$.

2. the SPO of $\succ$ implies that for every tuple o not in $w_{\succ}(r)$, there is a tuple $o' \in w_{\succ}(r)$ such that $o' \succ o$. Hence, if no edges going from $w_{\succ}(r)$ are contracted by P^-, every $o \notin w_{\succ}(r)$ will still be dominated according to $\succ'$.

3. Follows from the definition of the *prefix* contraction.

4. From 1 we know that $w_{\succ}(r) \subseteq w_{\succ'}(r)$. For every tuple $o \in w_{\succ'}(r) - w_{\succ}(r)$, at least one edge going to it in $\succ$ has been contracted by P^-. Thus, $w_{\succ'}(r) \subseteq w_{\succ}(r) \cup \sigma_{F_{E(P^-)}}(r)$ and $w_{\succ'}(r) = w_{\succ'}(w_{\succ}(r) \cup \sigma_{F_{E(P^-)}}(r))$. □

According to Proposition 6.4, the result of winnow under a contracted preference is always a superset of the result of winnow under the original preference. The second property shows when the contraction does not change the result of winnow. Running the winnow query is generally expensive, thus one can first evaluate $\sigma_{F_{S(P^-)}}$ (or $\sigma_{F_{S(CON)}}$, if P^- is a prefix contraction) over the computed result of the original winnow. If the result is empty, then computing the winnow under the contracted preference relation is not needed.

The last statement of the proposition is useful when the set r is large and thus running $w_{\succ'}$ over the entire set r is expensive. Instead, one can compute $\sigma_{F_{E(P^-)}}(r)$ and then evaluate $w_{\succ'}$ over $(w_{\succ}(r) \cup \sigma_{F_{E(P^-)}}(r))$ (assuming that $w_{\succ}(r)$ is already known).

EXAMPLE 6.15 *Let a preference relation $\succ$ be defined by $F_{\succ}(o,o') \equiv o.p < o'.p$, and a base contractor CON of $\succ$ be defined by $F_{CON}(o,o') \equiv o.p = 0 \wedge o'.p = 3$, where p is an Q-attribute. Take set of tuples $r = \{1,2,3,4\}$ in which every tuple has a single attribute p. Then $w_{\succ}(r) = \{1\}$. Take two minimal full contractors P_1^- and P_2^- of $\succ$ by CON defined by the following formulas*

$$F_{P_1^-}(o,o') \equiv o.p = 0 \wedge 0 < o'.p \leq 3$$
$$F_{P_2^-}(o,o') \equiv 0 \leq o.p < 3 \wedge o'.p = 3$$

The corresponding contracted preference relations $\succ_1$ and $\succ_2$ are defined by $F_{\succ_1}(o,o') \equiv F_{\succ}(o,o') \wedge \neg F_{P_1^-}(o,o')$ and $F_{\succ_2}(o,o') \equiv F_{\succ}(o,o') \wedge \neg F_{P_2^-}(o,o')$. The full contractor P_1^- is prefix, thus $F_{S(P_1^-)}(o) \equiv F_{S(CON)} \equiv o.p = 0$. The full contractor P_2^- is not prefix, and $F_{S(P_2^-)}(o) \equiv 0 \leq o.p < 3$.

First, $\sigma_{F_{S(P_1^-)}}(w_{\succ}(r)) = \emptyset$ implies $w_{\succ_1}(r) = w_{\succ}(r)$. Second, $\sigma_{F_{S(P_2^-)}}(w_{\succ}(r))$ is not empty and equal to $\{1\}$. Note that $F_{E(P_2^-)}(o) \equiv o.p = 3$. Hence, $\sigma_{F_{E(P_2^-)}}(r) = \{3\}$ and $w_{\succ_2}(r) = w_{\succ_2}(w_{\succ}(r) \cup \sigma_{F_{E(P_2^-)}}(r)) = \{1,3\}$.

6.10 Experimental evaluation

In this section, we present the results of an experimental evaluation of the preference contraction framework proposed here. We implemented the following operators of preference contraction: prefix contraction (denoted as PREFIX), preference-protecting minimal contraction (P^+-MIN), meet contraction (MEET), and preference-protecting meet contraction (P^+-MEET). PREFIX was implemented using Algorithm 6.3, P^+-MIN according to Theorem 6.4, MEET according to Theorem 6.5, and P^+-MEET according to Theorem 6.6. We used these operators to contract finite preference relations stored in a database table $R(X,Y)$. The preference relations used in the experiments were finite *skyline preference* *relations* [BKS01]. Such relations are often used in database applications. We note that such relations are generally not materialized (as database tables) when querying databases with skylines. However, they may be materialized in scenarios of preference elicitation [BGL07]. To generate such relations, we used the NHL 2008 Player Stats dataset [nhl08] of 852 tuples. Each tuple has 18 different attributes out of which we used 5. All algorithms used in the experiments were implemented in Java 6. We ran the experiments on Intel Core 2 Duo CPU 2.1 GHz with 2.0 GB RAM. All tables were stored in a PostgreSQL 8.3 database.

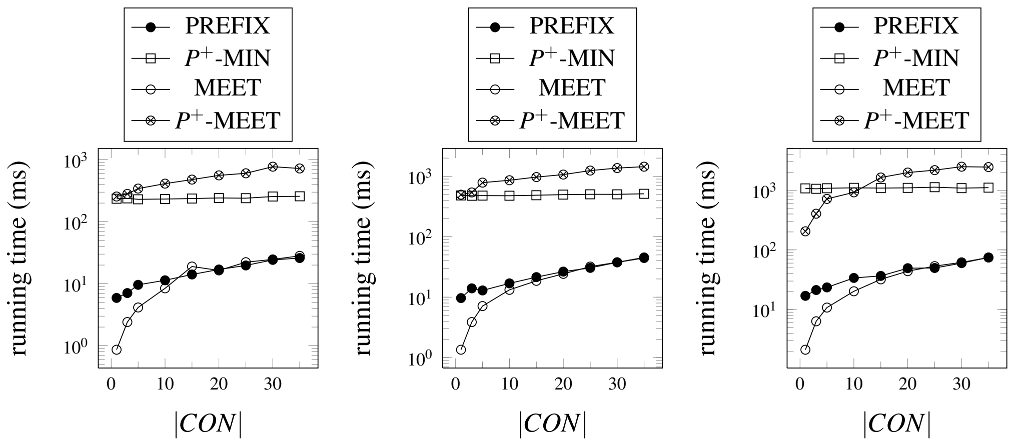
We have carried out two sets of experiments with the preference contraction algorithms. In the first set, we model the scenario when base contractors are manually constructed by user. Thus, we assume that such base contractors are of comparatively small size. In the second set of experiments, we assume that base contractors are constructed automatically and hence may be of large size.

Base contractors of small size

The aim of the experiments shown here is twofold. First, they demonstrate that the algorithms of preference contraction we have proposed are of high performance and scale well with the size of contracted preference relations (given base contractors of small size). Second, they show that when a base contractor is of small size, the difference between the sizes of full contractors computed by different algorithms may be significant. It implies that in real life applications, an appropriate contraction algorithm needs to be selected carefully depending on the required semantics.

Preference relations we use here consist of 2000, 3000, and 5000 edges. The sizes of base contractors range from 1 to 35 edges. We do not pick more than 35 edges assuming that in this scenario a user unlikely provides a large set of preferences to discard. For every base contractor size, we randomly generated 10 different base contractors and computed the average time spent to compute full contractors and the average size of them. The relations P^+ storing preferences to protect contained 25% of edges of the corresponding preference relation.

Figure 6-11 shows how the running times of contraction operators depend on the size of a preference relation to contract and the size of a base contractor. As we can observe, PREFIX has the best performance among all operators, regardless of the size of the preference relation and the base contractor relation. Note also that the running times of preference-protecting operators are significantly larger then the running times of their unconstrained counterparts. These running times predominantly depend on the time spent to compute the transitive closure of P^+.

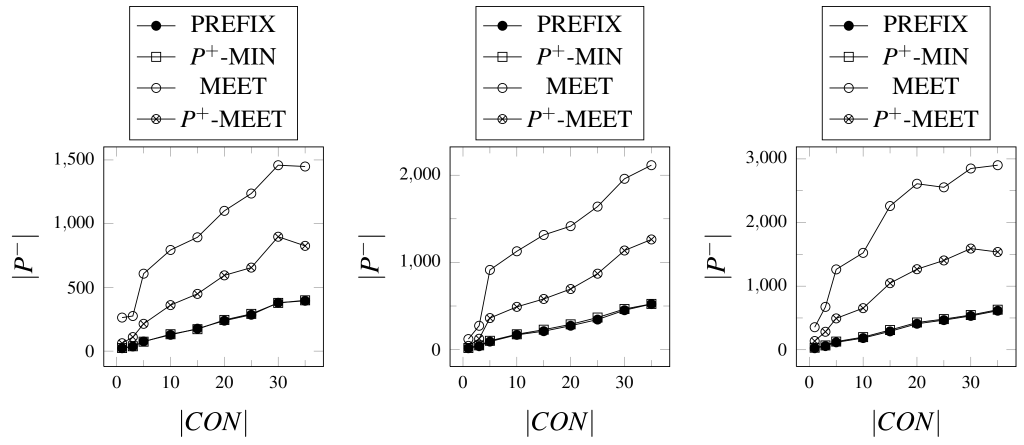


(a) $|\succ| = 2000$ (b) $|\succ| = 3000$ (c) $|\succ| = 5000$

Figure 6-11: *Contraction performance. Small base contractors*

Figure 6-12 shows the dependency of the full contractor size on the size of preference relation and the size of base contractor. For every value of the base contractor size, the charts show the average size of the corresponding full contractor. Notice that the preference protection constraint does not affect much the sizes of full contractors computed by PREFIX and P^+-MIN – they almost always coincide. That is due to the fact that even though the full contractors computed by these algorithms have different properties, they are both *minimal*. At the same time, the sizes of full contractors computed by MEET and P^+-MEET differ significantly – the size of a full meet contractor is generally twice as large as the size of the corresponding full P^+-protecting meet contractor. This is justified by the semantics of those full contractors: a full meet contractor is a union of all minimal full contractors of $\succ$ by CON, while a full P^+-protecting meet contractor is its subset.

Another important observation is that the sizes of minimal full contractors (PREFIX and P^+-MIN) and full meet contractors (MEET and P^+-MEET) differ significantly. Hence, the constraint of minimality has a high effect on the sizes of full contractors when base contractors are of small size.

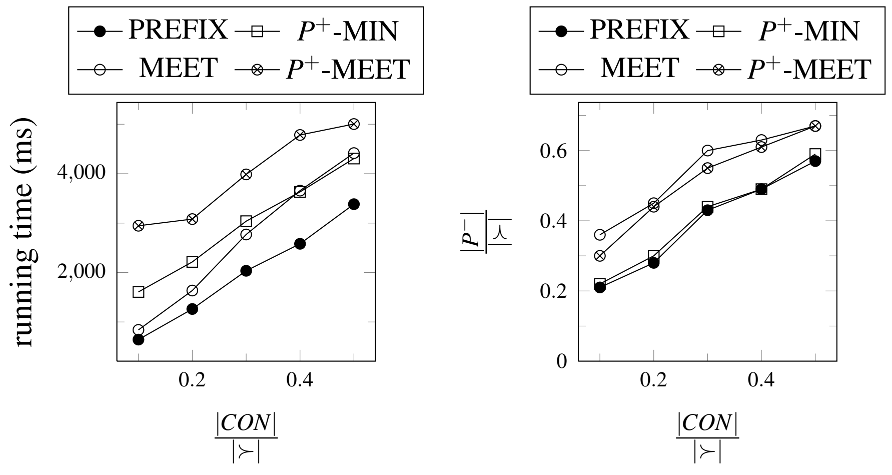


(a) $|\succ| = 2000$ (b) $|\succ| = 3000$ (c) $|\succ| = 5000$

Figure 6-12: *Full contractor size. Small base contractors*

Base contractors of large size

The aim of the next set of experiments is to demonstrate that the algorithms of preference contraction proposed here scale well with the size of preference relation to contract as well as the size of base contractor. We also show here that when a base contractor is large and consists of similar edges, the differences in size of minimal full contractors and meet full



(a) Contraction performance (b) Full contractor size

Figure 6-13: *Large base contractors*

contractors becomes insignificant. Hence, semantical differences of preference contraction operators are less important in such cases.

In contrast to the previous set of experiments, base contractors used here are of large size. We construct them from *similar edges*. Two edges xy and $x'y'$ are considered similar if the tuples x, x' and y, $'y$ are similar. We use the cosine similarity measure to compute similarity of tuples. Here we fixed the size of the preference relation to 5000. The sizes of base contractors range from 10% to 50% of preference relation size. The size of every P^+ is 25% of the corresponding preference relation size. Similarly to the previous experiment, we computed the performance of the contraction operators and the sizes of generated full contractors. The results are shown in Figure 6-13.

An important observation here is that even though meet full contractors take more time to compute than minimal full contractors, the difference between the running times is not as large as in the previous experiments. Notice also that the difference in size of all full contractors computed in this experiment is also small. Hence, we can conclude that in the case of massive preference contraction (i.e., when CON is of large size), the contraction operation selected to perform that change matters less than in the case of a small change (as in the previous experiment).

Second, consider the value of

$$aux(CON, P^-) = \frac{|P^-|}{|CON|} - 1$$

in this and the previous experiment. $aux(CON, P^-)$ is equal to the average number of edges contracted to contract one edge of CON. Due to the similarity of edges in P^-, the value of $aux(CON, P^-)$ is significantly smaller in this experiment than in the previous one. In Figure 6-12(c), $aux(CON, P^-)$ ranges from 16 to 23 for PREFIX. In contrast, $aux(CON, P^-)$ according to Figure 6-13(b) ranges from to 1.1 to 0.1 for the same algorithm. Low values of $aux(CON, P^-)$ here are due to the fact that similar preferences to contract have many *common reasons for holding them*. Moreover, the fact that the sizes of minimal and meet full contractors are of similar sizes implies that CON containing similar edges already contains many *reasons for not holding them*.

Note that in all experiments, the time spent to compute any full contractor did not go beyond 5 seconds. If the base contractor is small and preference protection is not used, then these times are even less than 100ms. Hence we conclude that the algorithms we proposed to contract finite relations are efficient and may be used in real-life database applications.

6.11 Related work

Relationships with other operators of preference relation change

A number of operators of preference relation change have been proposed so far. An operator of preference revision is defined in [Cho07b]. A preference relation there is *revised* by another preference relation called a *revising relation*. The result of revision is still another preference relation. [Cho07b] defines three semantics of preference revision – union, prioritized, and Pareto – which are different in the way an original and a revising preference relations are composed. For all these semantics, [Cho07b] identifies cases (called 0-, 1-*, and* 2-*conflicts*) when the revision fails, i.e., when there is no SPO preference relation satisfying the operator semantics. This work considers revising preference relations only by preference relations. Although it does not address the problem of discarding subsets of preference relations explicitly, revising a preference relation using Pareto and prioritized revision operators may result in discarding a subset of the original preference relation. It has been shown here that the revised relation is an SPO for limited classes of the composed relations.

Another operator of preference relation change is defined in [BGS06]. This work deals with a special class of preference relations called *skyline* [BKS01]. Preference relations in [BGS06] are changed by *equivalence relations*. In particular, a modified preference relation is an extension of the original relation in which specified tuples are *equivalent*. This change operator is defined for only those tuples which are *incomparable* or already *equivalent* according to the original preference relation. This preference change operator only adds new edges to the original preference relation, and thus, preference relation contraction cannot be expressed using this operator.

In [MC08], we introduced the operation of minimal preference contraction for preference relations. We studied properties of this operation and proposed algorithms for computing full contractors and preference-protecting full contractors for finitely stratifiable base contractors. In the current work, we introduce the operations of meet and meet preference-protecting contraction, and propose methods for computing them. We also provide experimental evaluation of the framework and a comprehensive discussion of related work.

Relationships with the belief revision theory

Preferences can be considered as a special form of human *beliefs*, and thus their change may be modeled in the context of the belief change theory. The approach here is to represent beliefs as truth-functional logical sentences. A *belief set* is a set of the sentences that are believed by an agent. A common assumption is that belief sets are closed under logical consequence. The most common operators of belief set change are *revision* and *contraction* [AGM85]. A number of versions of those operators have been proposed [Han98] to capture various real life scenarios.

This approach is quite different from the preference relation approach. First, the language of truth functional sentences is rich and allows for rather complex statements about preferences: conditional preferences ($a > b \rightarrow c > d$), ambiguous preferences ($a > b \vee c > d$) etc. In contrast to that, preferences in the preference relation framework used in the current work are certain: given a preference relation $\succ$, it is only possible to check if a tuple is preferred or not to another tuple. Another important difference of these two frameworks is that the belief revision theory exploits the open-world assumption, while the preference relation framework uses the closed-world assumption. In addition to that, belief revision is mostly studied in the context of finite domains. However, the algorithms we have proposed here can be applied to finite and infinite preference relations.

Relationships with the preference state framework

Another preference representation and change framework close to the belief revision theory is the *preference state* framework [Han95]. As in belief revision, a preference state is a logically closed sets of sentences describing preferences of an agent. However, every preference state has an underlying set of preference *relations*. The connection between states and relations is as follows. A preference relation (which is an order of tuples) is an unambiguous description of an agent preference. A preference relation induces a set of logical sentences which describe the relations. However, it is not always the case that people's preferences are unambiguous. Hence, every preference *state* is associated with a *set* of possible preference relations.

Here we show an adaptation of the preference state framework to the preference relation framework. As a result, we obtain a framework that encompasses preference contraction and restricted preference revision.

DEFINITION 6.9 *An* alternative *is an element of $\mathcal{U}$. Nonempty subsets of $\mathcal{U}$ are called* sets of alternatives. *The tuple language $\mathcal{L}_{\mathcal{U}}$ is defined as*

- *if $X, Y \in \mathcal{U}$ then $X > Y \in L_{\mathcal{U}}$*
- *if $X > Y \in L_{\mathcal{U}}$ then $\neg(X > Y) \in L_{\mathcal{U}}$.*

A subset of $\mathcal{L}_{\mathcal{U}}$ is called a *restricted preference set*. The language defined above is a very restricted version of the language in [Han95] since the only Boolean operator allowed is negation. Throughout the discussion, we assume that the set of alternatives is fixed to a subset $\mathcal{U}_r$ of $\mathcal{U}$.

DEFINITION 6.10 *Let R be a subset of $\mathcal{U}_r \times \mathcal{U}_r$. The set $[R]$ of sentences is defined as follows:*

- *$x > y \in [R]$ iff $xy \in R$*
- *$\neg(x > y) \in [R]$ iff $x, y \in \mathcal{U}_r$ and $x > y \notin [R]$*

DEFINITION 6.11 *A binary relation $R \subset \mathcal{U}_r \times \mathcal{U}_r$ is a* restricted preference model *iff it is a strict partial order. Given a restricted preference model R, the corresponding $[R]$ is called a* restricted preference state.

In contrast to the definition above, the preference model in [Han95] is defined as a *set* of SPO relations, and a preference state is an intersection of $[R]$ for all members R of the corresponding preference model.

We define two operators of change of restricted preference states: revision and contraction. Restricted states here are changed by sets of statements. In [Han95], a change of a preference state by a set of sentences is defined as the corresponding change by the conjunction of the corresponding statements. Moreover, change by any set of sentences is allowed. In the adaptation of that framework we define here, conjunctions of statements are not a part of the language. Moreover, preference revision [Cho07b] only allows for adding new preferences, and preference relation contraction we have proposed in this work allows only discarding existing preferences. Here we aim to define the operator of restricted preference set revision which captures the semantics of latter operator.

DEFINITION 6.12 *A restricted preference set S is called* positive *iff all sentences it contains are in form $A > B$ for some $A, B \in \mathcal{U}_r$. Analogously, S is* negative *iff it only contains sentences in form $\neg(A > B)$ for some $A, B \in \mathcal{U}_r$.*

A restricted preference set is a complement *of S (denoted as $\overline{S}$) if for all $A, B \in \mathcal{U}_r$, $A > B \in S$ iff $\neg(A > B) \in S$ and $\neg(A > B) \in S$ iff $A > B \in S$.*

A relation R_S is a minimal representation *of a restricted preference state S iff R_S is a minimal relation such that $S \subseteq [R_S]$.*

Positive and negative restricted preference sets are used to change restricted preference states. Intuitively, a positive preference set represents the existence of preferences while a negative set represents a lack of preferences.

DEFINITION 6.13 *Let R be a restricted preference model. Then the operator $*$ on R is a* restricted preference revision on *R if and only if for all positive/negative restricted preference sets S, $R * S = \cap\{R'\}$ for all R' such that*

1. *$S \subseteq [R']$*

2. *R' is an SPO*

3. *there is no SPO R'' with $S \subseteq [R'']$ such that $R \subseteq R'' \subset R'$ (if S is positive) or $R' \subset R'' \subseteq R$ (if S is negative).*

The last condition in the definition above expresses the minimality of restricted preference state change. This condition is different for positive and negative sets: when we add positive statements, we do not want to discard any existing positive sentences, and when negative statements are added, no new positive sentences should be added. The restricted preference revision operator defined above is different from preference state revision in [Han95]. First, preference state revision allows for revision by (finite) sets of arbitrary sentences, not only positive and negative sentences, as here. Second, the minimality condition here is defined using set containment while in [Han95] it is defined as a function of symmetric set difference of the original preference relations and R'. As a result, revising a preference state by a positive/negative sentence may result in losing an existing positive/negative sentence. The last difference is based on preference state representation: the result of preference revision in [Han95] is a union of relations R'' while in our case it is an intersection.

Below we define the operator of contraction for restricted preference states which is similar to the contraction of preference states.

DEFINITION 6.14 *Let R be a restricted preference model. Then the operator $\div$ on R is* restricted preference contraction on *R if and only if for all positive/negative restricted preference sets S, $R \div S = R * \overline{S}$.*

Given the operators on restricted preference states we have defined here, their relationships with the preference relation change framework are straightforward.

PROPOSITION 6.5 *Let R be a restricted preference model, S be a positive or negative restricted preference set, and R_S be a minimal representation of S. Then $R * S$ is*

1. *$\emptyset$, if S is a positive restricted preference set and $R \cup R_S$ has a cyclic path,*

2. *$TC(R \cup R_S)$, if S is a positive restricted preference set and $R \cup R_S$ has no cyclic paths,*

3. *$\cap\{R - P^- \mid P^-$ is a minimal full contractor of R by $R_{\overline{S}}\}$, if S is a negative restricted preference set,*

where TC is the transitive closure operator.

PROOF

When a restricted preference model is revised by a positive preference set, the resulting relation $R * S$ is the intersection of all minimal SPO extensions R' of R and R_S (i.e., R' has to contain an edge from A to B if $A > B \in S$). Such an extension R' does not exist if there is an cyclic path in $R \cup R_S$. However, if no cyclic paths exist, then there is only one such a minimal extension R' which is equal to the transitive closure of $R \cup R_S$. Hence, $R * S = TC(R \cup R_S)$. We note that this result is equivalent to the result of the *union preference revision* [Cho07b].

When a restricted preference model is revised by a negative preference set, the resulting relation $R * S$ has to be a subset of R. Moreover, for all $\neg(A > B) \in S$, there should be no edge from A to B in $R * S$. Hence, $R * S$ is an intersection of minimally contracted R by $R_{\overline{S}}$, which is a result of the full meet contraction of R by $R_{\overline{S}}$. □

Below we list some properties of the revision and contraction operators of restricted preference states.

PROPOSITION 6.6 *Let R be a restricted preference model and S be a positive/negative restricted preference set. Then*

1. *$R * S$ is an SPO (closure)*

2. *$S \subseteq [R * S]$ unless S is positive and $R_S \cup R$ has a cyclic path (limited success)*

3. *If $S \subseteq [R]$, then $R = R * S$ (vacuity)*

PROOF

All the properties here follow from Proposition 6.5. Namely, property 1 follows from the fact that the result of $R \ ^* \ S$ is an SPO in every case of Proposition 6.5. Property 2 follows from Proposition 6.5 and the definition of $[R \ ^* \ S]$. Property 3 follows from Proposition 6.5 and 1) $S \subseteq [R]$ implies $R_S \subseteq R$ (if S is positive), and 2) a minimally contracted preference relation is equal to itself if contracted by non-existent edges (if S is negative). □

PROPOSITION 6.7 *Let R be a restricted preference model and S be a restricted positive/negative preference set. Then*

1. *$R \div S$ is an SPO (closure)*
2. *$S \subseteq [R \div S]$ unless S is negative and $R_{\overline{S}} \cup R$ has a cyclic path (limited success)*
3. *If $S \cap [R] = \emptyset$, then $R = R \div S$ (vacuity)*
4. *$R \ ^* \ S = (R \div \overline{S}) \ ^* \ S$ unless S is positive and $R_S \cup R$ has a cyclic path (limited Levi identity)*
5. *$R \div S = R \ ^* \ \overline{S}$ (Harper identity, by definition)*

PROOF

Properties 1, 2, and 3 follow from Proposition 6.6. Property 4 follows from the fact that $R \div \overline{S} = R \ ^* \ S$ by definition, and Proposition 6.6 implies $R \ ^* \ S = (R \ ^* \ S) \ ^* \ S$ when either S is negative or S is positive but $R_S \cup R$ has no cyclic path. □

An important difference between the restricted preference-set change operators and the corresponding change operators from [Han95] is that the restricted versions are not always successful (property 2 in Proposition 6.5), and Levi identity holds for a certain class of restricted preference sets. In addition to that, the operator of preference set contraction in [Han95] has the property of inclusion ($R \subseteq R \div S$) and recovery (if $S \subseteq [R]$, then $R = (R \div S) \ ^* \ S$). As for the restricted framework defined here, inclusion does not hold due to the representation of a preference model as a single SPO relation. Recovery does not hold here due to the restrictions to the language (namely, not allowing disjunction of sentences).

We note that one of the main targets of our current work was development of an efficient and practical approach of contracting preference relations in the binary relation framework, in the finite and the finitely representable cases. In addition to the defining semantics of preference contraction operators, we have also developed a set of algorithms which can be used to compute contractions. We have tested them on real-life data and demonstrated their efficiency. In contrast, [Han95] focuses more on semantical aspects of preference change and does not address computational issues of preference change operators. In particular, finite representability is not addressed.

Relationship with other frameworks

An approach of preference change is proposed in [CP06]. Preferences here are changed via *interactive example critiques*. This paper identified three types of common critique models: similarity based, quality based, and quantity based. However, no formal framework is provided here. [Fre04] describes revision of *rational* preference relations over propositional formulas. The revision operator proposed here satisfies the postulates of success and minimal change. The author shows that the proposed techniques work in case of revision by a single statement and can be extended to allow revisions by multiple statements.

[DLSL99] proposes algorithms of incremental maintenance of the transitive closure of graphs using relational algebra. The graph modification operations are edge insertion and deletion. Transitive graphs in [DLSL99] consist of two kinds of edges: the edges of the original graph and the edges induced by its transitive closure. When an edge xy of the original graph is contracted, the algorithm also deletes all the transitive edges uv such that all the paths from u to v in the original graph go through xy. As a result, such contraction is not minimal according to our definition of minimality. Moreover, [DLSL99] considers only finite graphs, whereas our algorithms can work with infinite relations.

Chapter 7

Conclusions and future work

In this work, we researched on the following aspects of preference construction for database querying: preference specification, discovery of preferences based on user feedback, and construction of preferences by their change.

7.1 Preference specification

Exploring this direction, we proposed two frameworks of preference specification: *p-skylines* and *HCP-nets*.

Preference relations in the p-skyline framework are constructed by specifying three characteristics of preferences: relevant attributes, preferences over these attributes, and relative importance of the attributes. Relative importance of attributes in this model is captured using directed graphs over attributes – *p-graphs*. We showed that the p-skyline framework can be used in the context of the *binary relation preference framework*. In particular, we proved that p-skyline relations are representable as *finite preference formulas*. Hence, numerous methods of query evaluation and optimization developed in the binary relation preference framework are applicable to p-skyline preference queries, too. We also showed that the p-graph notation for p-skyline relations can be used to perform essential operations with such relations. In particular, the problems of checking equivalence and containment of (infinite) p-skyline relation may be reduced to the problems of checking the corresponding relationships between their p-graphs, which are finite relations. We also showed that the dominance testing in the p-skyline framework can be efficiently performed using p-graphs.

Another problem we studied in the context of the p-skyline framework is computation of minimal extensions of a p-skyline relation. This problem generally arises in cases when one needs to check if a given p-skyline relation is *maximal* (given a set of relevant attributes and atomic preferences over them) in which certain preferences do not hold. We proposed a set of rewriting rules which allow to compute *all* minimal extensions of a p-skyline relation. The rules operate on the expression defining a p-skyline relation and can be used to compute all its minimal extensions in polynomial time.

As we showed here, the p-skyline framework naturally fits two widely used approaches of preference specification: skylines and preference constructors. That implies that methods of evaluation and optimization of preference queries in these frameworks may potentially be adapted to p-skyline preference queries. We also provided an example of such adaptation – we showed how an efficient skyline algorithm SFS can be used to evaluate p-skyline queries.

The other framework we proposed here to specify preferences in the binary relation preference framework is *HCP-nets*. It is a variant of the graphical preference specification approach of *CP-nets* widely used in the area of Artificial Intelligence. In HCP-nets, we addressed some semantical and syntactical issues of CP-nets.

Similarly to CP-nets, preferences in HCP-nets are specified using graphs over attributes whose edges capture *conditionality* of preferences. In contrast to CP-nets which follow the *ceteris paribus* (everything else is equal) semantics, HCP-nets exploit the *everything else except descendants is equal* semantics. As we showed in the current work, the latter semantics not only captures possible reasoning for conditionality of preferences, but also brings computational benefits. In particular, the methods of dominance testing we proposed here for HCP-nets may be implemented in polynomial time. At the same time, dominance testing for CP-nets is intractable in general.

In contrast to CP-nets which are defined for finite domains, HCP-nets may be used with finite as well as infinite domains, which makes this model compatible with the binary relation preference framework. Moreover, we showed here that preference relations induced by HCP-nets are representable as preference formulas of *polynomial size*, which may be potentially used as a foundation for efficient methods of querying databases with HCP-nets. However, as our experiments showed, even unoptimized HCP-net preference queries may be evaluated much more efficiently than the corresponding CP-net preference queries.

In the context of the proposed frameworks of preference specification, we envision the following directions of future work.

In our study of the p-skyline framework, we mostly focused on the class of *full* p-skyline relations – those which are constructed on top of a *fixed* set of relevant attributes and such that each atomic preference is used exactly once in the definition of a p-skyline relation. An interesting direction here is to allow for variable sets of relevant attributes and use each atomic preference an arbitrary number of times when defining a p-skyline relation. Such an extension has a higher expressive power and is analogous to the *subspace skyline* [PJET05] extension of the skyline framework. Some challenging problems here are the semantics and methods of construction of p-graphs for such relations. Another interesting question is whether p-graphs in such an extension may be used to perform essential operations with preference relations, as in the p-skyline framework.

The attribute importance relation describing every p-skyline relation has the following property: if an attribute A is less important than a set of attributes S, then A is less important than *every* member of S. However, while studying properties of p-skyline relations, we observed that preference relations obtained as *intersections* of certain p-skyline relations may not have the property above. Namely, A may be less important than S but *not* less important than every member of S. Clearly, such attribute importance relationships may hold in real life. For example, one may feel that benefits in year and color of a car separately may not be more important than loses in its price, while benefits in *both* year and color may outweigh a higher price. It seems intuitive to consider such attribute importance as a relation over *sets* of attributes rather than distinct attributes, as in the p-skyline framework. Hence, an interesting research problem is to extend the p-skyline framework to allow specification of attribute importance as a relation over attribute sets.

In order to adapt CP-nets to the binary relation preference framework, we proposed to use HCP-nets whose semantics is different from the one of CP-nets. That semantic difference allowed us to develop an efficient method of constructing polynomial size preference formulas for HCP-nets. However, the question whether CP-nets (following the original semantics) defined on infinite domains are representable as polynomial size preference formulas remains open.

In our experiments, we showed that the performance of HCP-net preference queries is much higher than of CP-net preference queries. However, our experiments did not demonstrate if the semantic difference between the models affects such query results. Namely, when comparing the performance of CP-net and HCP-net preference queries, we used *small attribute domains*, because our large domain CP-net queries did not terminate in reasonable time. The results of such small-domain queries were too small (less than ten tuples) to draw any conclusions regarding the semantic effect on them. Hence, it is still an open problem if one can substitute CP-net queries (due to their low performance) with the corresponding HCP-net queries without any or with a little impact on query results. To research on that problem, one may have to carry out experiments on large domains with classes of CP-nets for which dominance testing is known to be in PTIME [BBD$^+$04].

In the comparison of HCP-nets and p-skyline relation in Section 5, we pointed out to important similarities between these two frameworks: i) an attribute in a p-graph of a p-skyline relation and in a conditional preference graph of an HCP-net is less important than all its descendents, and ii) a p-skyline relation may be defined similarly to the order induced by an HCP-net. This brings an interesting problem of generalizing these two frameworks into one and studying its properties.

7.2 Preference discovery

In our research on preference discovery, we focused on the problem of discovery of *attribute importance* using user feedback. We studied this problem in the context of the p-skyline framework. The following scenario of preference discovery was used. Given a set of tuples describing real objects, a user selects two disjoins sets of tuples: *superior examples* (those which she confidently likes in the set) and *inferior examples* (those which she confidently dislikes in the set). We assumed that the attributes relevant to user preferences along with the atomic preferences over these attribute were known beforehand. The goal was to construct a p-skyline relation on top of the set of relevant attributes and the atomic preferences such that it *optimally* favors the given superior and disfavors the given inferior examples. As the optimality criterion, we used *the maximality* of p-skyline relation.

We considered several problems of preference discovery in this context. The first one is the problem of existence of a p-skyline relation favoring superior examples and disfavoring inferior examples. The second one is the problem of computation of such a p-skyline relation. The third one is the problem of computation of an *optimal* p-skyline relation favoring superior and disfavoring inferior examples. We showed that the first problem is NP-complete, and the last two are FNP-complete.

Due to the intractability of these problems, we considered restricted versions of them, in which a user provides only superior examples. We showed that these problems are computationally simpler than the unrestricted ones. First, we proved that the problem of existence of a p-skyline relation favoring a set of tuples can be reduced to the problem of computing a skyline and checking the containment of two sets of tuples, and hence is polynomially solvable. Second, we proposed a polynomial time algorithm for computing such a p-skyline relation (optimal or not). The algorithm transforms a given set of superior examples into a system of *negative* constraints and then uses the rules of *minimal p-skyline extension* to compute an optimal p-skyline relation satisfying the system of negative constraints. The experimental study of the proposed approach of p-skyline relation discovery shows its high scalability and accuracy.

We envision the following directions of future work in this context. Our experimental study of the discovery approach showed that the accuracy of the algorithm generally grows if a user provides more superior examples. However, to provide more superior examples, a user has to spend more time exploring the provided set of tuples which may be rather large. Even though we showed that instead of exploring the entire data set, a user may explore only its skyline, it only partially simplifies the problem because skylines are known to be of large size for large data sets and a large number of relevant attributes. We believe that in many cases, a small subset of a skyline may be exposed to a user without loosing the "expressiveness" of the entire skyline. It is clear that exposing a small skyline subset instead of the entire skyline would increase the practicality of the proposed discovery approach. Hence, there is a need in methods of extracting a small expressive subsets from a skyline. Some promising measures of expressiveness of a skyline tuple are its *representativeness* [LYZZ07] and its *entropy value* [GSG07], or the number of p-skyline relations favoring the tuple. This direction has to be thoroughly investigated.

Another related problem is estimation of the "quality" of a set of superior examples. While experimenting with the discovery approach, we found that some sets of superior examples have higher "quality" than other sets of a similar size. That is, given such a high "quality" set of superior examples, there are only a few optimal p-skyline relations favoring it. Hence, the accuracy of the p-skyline discovery on such a set would be potentially higher. Therefore, there is a need in methods of estimation of such "quality". Such an estimation may be used in an *interactive* scenario of selecting superior examples. Namely, a user explores a set of tuples and selects superior examples one by one. After the selection of a superior example, the system estimates the "quality" of the examples selected so far and lets the user know if more examples are needed, or it has enough information to accurately compute a favoring p-skyline relation. Moreover, in each iteration of the process, the system may drop the tuples which if selected as superior examples would not affect the "quality" of the already selected examples. That would result in reducing the search space of superior examples. Such interactive scenarios of superior example selection need further investigation.

We showed that p-skyline relation discovery using both superior and inferior examples is an intractable problem in general. However, we believe that using inferior along with superior examples would improve the accuracy of p-skyline relation discovery. Thus, the problem of designing heurisics of p-skyline relation discovery using both types of examples together requires further investigation.

In the algorithm of preference discovery we developed in this work, the only type of user feedback we used was superior examples. However, the algorithm does not use superior examples directly but transforms them into a system of *negative constraints*. A negative constraint expresses the fact that a tuple is not preferred to another according to the sought preference relation. As a result, if another type of user feedback can be expressed as a system of negative constraints, our algorithm of p-skyline relation discovery may be used to compute the corresponding preference relations. Thus, a challenging problem of future work is to study other types user feedback and their adaptiveness to the proposed discovery approach.

Another interesting problem here is to adapt the proposed discovery approach to HCP-nets. As we believe, the hardest problem in the context of HCP-nets will be the discovery of *conditionality* of preferences.

7.3 Preference change

In our research on constructing preference relations by their change, we focused on the operation of discarding subsets of preference relations – *preference contraction*. We considered two aspects of this operation: semantical and computational.

First, we addressed the semantical properties of preference contraction. All the contraction operators we developed here satisfy a common property of operators of preference change – they *preserve SPO axioms* of a modified preference relation. Thus, preference relations are closed under the operators of preference contraction. The fundamental operator of preference contraction we considered in this context is *minimal contraction*. In addition to the SPO axioms preservation, it satisfies another common property of preference change operators – *minimality of preference change*. That is, in order to contract a preference relation by its subset *CON* (called a *base contractor*), it discards a minimal superset (called a minimal *full contractor*) of *CON*, whose removal from the preference relation is needed to satisfy the SPO preservation property.

An important property of this operator is that a minimal full contractor of a preference relation by a base contractor is not unique in general, and the number of such full contractors may be even infinite for an infinite preference relation. An important question here is that if a certain base contractor is computed, how do we know that it represents the change the user actually wanted to perform. To address that issue and give a user more freedom do specify preference change more precisly, we proposed a constrained version of the minimal contraction operator called *preference-protecting minimal contractor*. That operator has an addition parameter – a subset P^+ of the original preference relation which has to be protected in the contracted preference relation. Such P^+ may be viewed as the set of preferences which are of the highest importance for a user.

In some cases, in order to specify the preferences which need to be protected after contraction or elaborate the contraction, a user may want to know which preferences *may* potentially be lost when discarding *CON*. In order to compute such preferences, we proposed the operators of meet and preference-protecting meet contraction. The corresponding operators result in discarding the union of all minimal contractors and minimal preference-protecting contractors, respectively.

The next problem we considered here was the evaluation of the proposed operators for finite as well as infinite preference relations. Here we restricted ourselves to the class of *finitely stratifiable* base contractors, which may be finite as well as infinite. First, we proposed an algorithm for computing a minimal full contractor by a finitely stratifiable base contractor and presented two its implementations: for finite relations represented as database tables and infinite relations represented as finite formulas. We showed that the implementation for finite relations has a polynomial runtime. The implementation for infinite relations extensively uses *quantifier elimination*. Second, we presented methods of evaluation of meet and preference-protecting meet contractions. Third, we performed an extensive experimental evaluation of the proposed operators, where we demonstrated differences in the semantics and the performance of the contraction operators.

There are several interesting directions of future work based on the work described here. The main focus of our study of computational issues of preference contraction was the class of finitely stratifiable base contractors. We note that every finite base contractor is finitely stratifiable, but not every infinite base contractor is. Hence, the problem of definability as well as computation of a finitely representable full contractor by an arbitrary base contractor is still open.

In the related work, we showed some relationships of the proposed contraction operators and some operators of *preference revision* developed in [Cho07b]. We pointed out that the preference revision operators are not always successful – they fail when composed preference relations (the *original* and the *revising* relations) have *conflicts*, i.e., when the relation resulting in a composition of the two relations has *cycles*. An interesting application of the proposed contraction operators is to design a *fail-safe* revision operator. As we see this problem, before computing the revision, such an operator has to discard a minimal subset of one of the relations being composed (original or revising) such that the preference relation resulting in their composition would not have conflicts.

Another challenging problem of future work is to develop a universal operator of preference change which would allow to contract and revise a preference relation simultaneously. It is not clear whether such an operator may be decomposed into sequential applications of the existing operators of revision and contraction.

An interesting research problem is the development of a special language of changing preference relations. Such a language should have to support the entire variety of the existing operators of preference revision and contraction, and be simple and intuitive at the same time.

Bibliography

[AGM85] Carlos E. Alchourron, Peter Gardenfors, and David Makinson. On the logic of theory change: Partial meet contraction and revision functions. *The Journal of Symbolic Logic*, 50(2):510–530, 1985.

[AHV95] Serge Abiteboul, Richard Hull, and Victor Vianu. *Foundations of Databases*. Addison Wesley, 1995.

[BBD$^+$04] C. Boutilier, R. Brafman, C. Domshlak, H. Hoos, and D. Poole. CP-nets: A tool for representing and reasoning with conditional ceteris paribus preference statements. *Journal of Artificial Intelligence Research*, 21:135–191, 2004.

[BBHP99] Craig Boutilier, Ronen I. Brafman, Holger H. Hoos, and David Poole. Reasoning with conditional ceteris paribus preference statements. In *Proceedings of the Fifteenth Conference on Uncertainty in Artificial Intelligence*, pages 71–80, Stockholm, Sweden, July 1999. Morgan Kaufmann.

[BD02] Ronen I. Brafman and Carmel Domshlak. Introducing variable importance tradeoffs into CP-nets. In *Proceedings of the Eighteenth Conference in*

Uncertainty in Artificial Intelligence, pages 69–76, Edmonton, Alberta, Canada, August 2002. Morgan Kaufmann.

[BD04] R. Brafman and Y. Dimopoulos. Extended semantics and optimization algorithms for CP-networks. *Computational Intelligence*, 2004.

[BG96] Fahiem Bacchus and Adam Grove. Utility independence in a qualitative decision theory. In *In Proceedings of KR'96*, pages 542–552. Morgan Kaufmann, 1996.

[BGL07] Wolf-Tilo Balke, Ulrich Güntzer, and Christoph Lofi. Eliciting matters - controlling skyline sizes by incremental integration of user preferences. In *DASFAA*, pages 551–562, 2007.

[BGS06] Wolf-Tilo Balke, Ulrich Guntzer, and Wolf Siberski. Exploiting indifference for customization of partial order skylines. In *Proceedings of the 10th International Database Engineering and Applications Symposium*, pages 80–88, Delhi, India, December 2006. IEEE Computer Society.

[BKS01] Stephan Börzsönyi, Donald Kossmann, and Konrad Stocker. The skyline operator. In *Proceedings of the 17th International Conference on Data Engineering*, pages 421–430, Washington, DC, USA, 2001. IEEE Computer Society.

[Bou02] Craig Boutilier. A POMDP formulation of preference elicitation problems. In *Eighteenth national conference on Artificial intelligence*, pages 239–246, Menlo Park, CA, USA, 2002. American Association for Artificial Intelligence.

[CET05] Chee Yong Chan, Pin-Kwang Eng, and Kian-Lee Tan. Stratified Computation of Skylines with Partially-Ordered Domains. In *SIGMOD Conference*, pages 203–214. ACM, June 2005.

[CGGL03] Jan Chomicki, Parke Godfrey, Jarek Gryz, and Dongming Liang. Skyline with presorting. In *Proceedings of the 19th International Conference on Data Engineering (ICDE)*, pages 717–816, Bangalore, fIndia, March 2003.

[Cho03] Jan Chomicki. Preference formulas in relational queries. *ACM Trans. Database Syst.*, 28(4):427–466, 2003.

[Cho07a] J. Chomicki. Semantic optimization techniques for preference queries. *Inf.Syst.(IS)*, 32(5):670–684, 2007.

[Cho07b] Jan Chomicki. Database querying under changing preferences. *Annals of Mathematics and Artificial Intelligence*, 50(1-2):79–109, 2007.

[CJT$^+$06] Chee Yong Chan, H. V. Jagadish, Kian-Lee Tan, Anthony K. H. Tung, and Zhenjie Zhang. Finding k-dominant skylines in high dimensional space. In *SIGMOD Conference*, pages 503–514, June 2006.

[CKP00] Urszula Chajewska, Daphne Koller, and Ronald Parr. Making rational decisions using adaptive utility elicitation. In *Proceedings of the Seventeenth National Conference on Artificial Intelligence*, pages 363–369, 2000.

[CLRS01] Thomas H. Cormen, Charles E. Leiserson, Ronald L. Rivest, and Clifford Stein. *Introduction to Algorithms, Second Edition*. The MIT Press, September 2001.

[CP06] Li Chen and Pearl Pu. Evaluating critiquing-based recommender agents. In *Proceedings of AAAI-2006*, Boston, Massachusetts, USA, July 2006. AAAI Press.

[CS05] Jan Chomicki and Joyce Song. Monotonic and nonmonotonic preference revision. *arXiv.org*, cs/0503092, March 2005.

[DFP03] Didier Dubois, Hélène Fargier, and Patrice Perny. Qualitative decision theory with preference relations and comparative uncertainty: an axiomatic approach. *Artif. Intell.*, 148(1-2):219–260, 2003.

[DGKT06] Gautam Das, Dimitrios Gunopulos, Nick Koudas, and Dimitris Tsirogiannis. Answering top-k queries using views. In *VLDB '06: Proceedings of the 32nd international conference on Very large data bases*, pages 451–462. VLDB Endowment, 2006.

[DJ07] Carmel Domshlak and Thorsten Joachims. Efficient and non-parametric reasoning over user preferences. *User Modeling and User-Adapted Interaction*, 17(1-2):41–69, 2007.

[DLSL99] G. Dong, L. Libkin, J. Su, and L.Wong. Maintaining the transitive closure of graphs in SQL. *Int. Journal of Information Technology*, 5:46–78, 1999.

[DMT05] Yannis Dimopoulos, Pavlos Moraitis, and Alexis Tsoukias. Extending variable importance in preference networks. In *IJCAI-05 Multidisciplinary Workshop on Advances in Preference Handling*, July 2005.

[Doy04] Jon Doyle. Prospects for preferences. *Computational Intelligence*, 20(2):111–136, 2004.

[EK06] M. Endres and W. Kießling. Transformation of TCP-net queries into preference database queries. In *Proceedings of the ECAI 2006 Multidisciplinary Workshop on Advances in Preference Handling*, Riva del Garda, Italy, August 2006.

[Fis70] P.C. Fishburn. *Utility Theory for Decision-Making*. John Wiley & Sons, New York, 1970.

[FLN01] Ronald Fagin, Amnon Lotem, and Moni Naor. Optimal aggregation algorithms for middleware. In *PODS '01: Proceedings of the twentieth ACM SIGMOD-SIGACT-SIGART symposium on Principles of database systems*, pages 102–113, New York, NY, USA, 2001. ACM.

[Fre04] Michael Freund. On the revision of preferences and rational inference processes. *Artificial Intelligence*, 152(1):105–137, 2004.

[GLTW05] Judy Goldsmith, Jérôme Lang, Miroslaw Truszczynski, and Nic Wilson. The computational complexity of dominance and consistency in CP-nets. In *Proceedings of the Nineteenth International Joint Conference on Artificial Intelligence*, pages 144–149, Edinburgh, Scotland, UK, July 2005. Professional Book Center.

[GSG05] Parke Godfrey, Ryan Shipley, and Jarek Gryz. Maximal vector computation in large data sets. In *Proceedings of the 31st International Conference on Very Large Data Bases*, pages 229–240, Trondheim, Norway, August 2005. ACM.

[GSG07] Parke Godfrey, Ryan Shipley, and Jarek Gryz. Algorithms and analyses for maximal vector computation. *VLDB Journal*, 16(1):5–28, 2007.

[GW05] Krzysztof Gajos and Daniel S. Weld. Preference elicitation for interface optimization. In *Proceedings of UIST '05*, pages 173–182, New York, NY, USA, 2005. ACM.

[Han95] S. O. Hansson. Changes in preference. *Theory and Decision*, 38(1):1–28, 1995.

[Han98] Sven Ove Hansson. *Handbook of Defeasible Reasoning and Uncertainty Management Systems*, volume 3: Belief Change, chapter Revision of belief sets and belief bases, pages 17–75. Kluwer Academic Publishers, Dordrecht, October 1998.

[HEK03] Stefan Holland, Martin Ester, and Werner Kießling. Preference mining: A novel approach on mining user preferences for personalized applications. In *Proceedings of the Seventh European Conference on Principles and Practice of Knowledge Discovery in Databases*, pages 204–216, Cavtat-Dubrovnik, Croatia, September 2003. Springer.

[HK05] Bernd Hafenrichter and Werner Kießling. Optimization of relational preference queries. In *Proceedings of the 16th Australasian database conference*, pages 175–184, Darlinghurst, Australia, 2005. Australian Computer Society, Inc.

[HRGM03] Peter Haddawy, Angelo Restificar, Benjamin Geisler, and John Miyamoto. Preference elicitation via theory refinement. *Journal of Machine Learning Research*, 4:2003, 2003.

[JPL$^+$08] Bin Jiang, Jian Pei, Xuemin Lin, David W. Cheung, and Jiawei Han. Mining preferences from superior and inferior examples. In *Proceedings of the 14th ACM SIGKDD International Conference on Knowledge Discovery and Data Mining*, pages 390–398. ACM, August 2008.

[Kie02] Werner Kießling. Foundations of preferences in database systems. In *Proceedings of 28th International Conference on Very Large Data Bases*, pages 311–322, Hong Kong, China, August 2002. Morgan Kaufmann.

[Kie04] Werner Kießling. Preference constructors for deeply personalized database queries. Technical report, Institute of Computer Science, University of Augsburg, March 2004.

[Kie05] W. Kießling. Preference queries with SV-semantics. *11th International Conference on Management of Data (COMAD 2005)*, pages 15–26, 2005.

[KK02] Werner Kießling and Gerhard Köstler. Preference SQL - Design, Implementation, Experiences. In *Proceedings of 28th International Conference on Very Large Data Bases*, pages 990–1001, Hong Kong, China, August 2002. Morgan Kaufmann.

[KKR95] Paris C. Kanellakis, Gabriel M. Kuper, and Peter Z. Revesz. Constraint query languages. *Journal of Computer and System Sciences*, pages 26–52, 1995.

[KRR02] Donald Kossmann, Frank Ramsak, and Steffen Rost. Shooting Stars in the Sky: An Online Algorithm for Skyline Queries. In *VLDB*, pages 275–286. Morgan Kaufmann, August 2002.

[LHL97] Greg Linden, Steve Hanks, and Neal Lesh. Interactive assessment of user preference models: The automated travel assistant. In *In Proceedings of the Sixth International Conference on User Modeling*, pages 67–78. Springer, 1997.

[LwYwH$^+$08] Jongwuk Lee, Gae won You, Seung won Hwang, Joachim Selke, and Wolf-Tilo Balke. Optimal preference elicitation for skyline queries over categorical domains. In *Proceedings of the 19th International Conference on Database and Expert Systems Applications*, pages 610–624, Turin, Italy, September 2008. Springer.

[LwYwH09] Jongwuk Lee, Gae won You, and Seung won Hwang. Personalized top-k skyline queries in high-dimensional space. *Inf. Syst.*, 34(1):45–61, 2009.

[LYZZ07] Xuemin Lin, Yidong Yuan, Qing Zhang, and Ying Zhang. Selecting stars: The k most representative skyline operator. In *ICDE*, pages 86–95. IEEE, April 2007.

[LZLL07] Ken C. K. Lee, Baihua Zheng, Huajing Li, and Wang-Chien Lee. Approaching the Skyline in Z order. In *VLDB*, pages 279–290. ACM, September 2007.

[MC07] Denis Mindolin and Jan Chomicki. Hierarchical CP-networks. In *Proceedings of the Third Multidisciplinary Workshop on Advances in Preference Handling (M-PREF)*, Vienna, Austria, September 2007.

[MC08] Denis Mindolin and Jan Chomicki. Minimal contraction of preference relations. In *Proceedings of AAAI-2008*, pages 492–497, Chicago, Illinois, USA, July 2008. AAAI Press.

[MC09] Denis Mindolin and Jan Chomicki. Discovering relative importance of skyline attributes. *to appear in the Proceedings of the VLDB Endowment*, August 2009.

[MD02] Michael McGeachie and Jon Doyle. Efficient utility functions for ceteris paribus preferences. In *Eighteenth national conference on Artificial intelligence*, pages 279–284, Menlo Park, CA, USA, 2002. American Association for Artificial Intelligence.

[MPJ07] Michael D. Morse, Jignesh M. Patel, and H. V. Jagadish. Efficient skyline computation over low-cardinality domains. In *VLDB*, pages 267–278, Vienna, Austria, September 2007. ACM.

[nhl08] NHL.com Player Stats, 2008. `http://www.nhl.com/ice/playerstats.htm`.

[Pap94] Christos M. Papadimitriou. *Computational complexity*. Addison-Wesley, Reading, Massachusetts, 1994.

[PJET05] Jian Pei, Wen Jin, Martin Ester, and Yufei Tao. Catching the Best Views of Skyline: A Semantic Approach Based on Decisive Subspaces. In *VLDB*, pages 253–264. ACM, August 2005.

[Sch03] Bernd Schröder. *Ordered Sets: An Introduction*. Birkhäuser Boston, Boston, MA, 2003.

[SL01] Sybil Shearin and Henry Lieberman. Intelligent profiling by example. In *IUI '01: Proceedings of the 6th international conference on Intelligent user interfaces*, pages 145–151, New York, NY, USA, 2001. ACM.

[TEO01] Kian-Lee Tan, Pin-Kwang Eng, and Beng Chin Ooi. Efficient Progressive Skyline Computation. In *VLDB*, pages 301–310. Morgan Kaufmann, September 2001.

[VFP06] Paolo Viappiani, Boi Faltings, and Pearl Pu. Preference-based search using example-critiquing with suggestions. *J. Artif. Intell. Res. (JAIR)*, 27:465–503, 2006.

[VH99] Peter Haddawy Vu Ha. A hybrid approach to reasoning with partial preference models. In *In Proceedings of the Fifteenth Conference on Uncertainty in Artificial Intelligence*, pages 263–270, 1999.

[Wil04] Nic Wilson. Extending CP-nets with stronger conditional preference statements. In *Proceedings of the Nineteenth National Conference on Artificial*

Intelligence, Sixteenth Conference on Innovative Applications of Artificial Intelligence, pages 735–741, San Jose, California, USA, July 2004. AAAI Press / The MIT Press.

[Wil06] Nic Wilson. An efficient upper approximation for conditional preference. In *European Conference on Artificial Intelligence (ECAI)*, pages 472–476. IOS Press, August 2006.

[Yd07] Fusun Yaman and Marie desJardins. More-or-less CP-networks. In *Proceedings of the Twenty-Third Conference on Uncertainty in Artificial Intelligence*. AUAI Press, July 2007.

[YLL$^+$05] Yidong Yuan, Xuemin Lin, Qing Liu, Wei Wang, Jeffrey Xu Yu, and Qing Zhang. Efficient computation of the skyline cube. In *VLDB*, pages 241–252. ACM, August 2005.

Appendix A

Omitted proofs

THEOREM 3.1. *For every $\succ \in \mathcal{F}_{\mathcal{H}}$, there exists a $(W, \mathcal{H})$-structure which induces a relation $\succ_{(W,\mathcal{H})}$ equivalent to $\succ$. Moreover,*

1. *if $\succ$ is induced by an atomic preference $>_A$, then $W_A = \emptyset$*

2. *if $\succ = \succ_1 \otimes \succ_2$, then*

$$W_A = \begin{cases} W_A^1, & \text{if } A \in Var(\succ_1) \\ W_A^2, & \text{if } A \in Var(\succ_2) \end{cases}$$

3. *if $\succ = \succ_1 \ \& \ \succ_2$, then*

$$W_A = \begin{cases} W_A^1 \cup Var(\succ_2), & \text{if } A \in Var(\succ_1) \\ W_A^2, & \text{if } A \in Var(\succ_2) \end{cases}$$

for $(W^1, \mathcal{H})$ and $(W^2, \mathcal{H})$ inducing relations equivalent to $\succ_1$ and $\succ_2$.

PROOF

We show here that for any $\succ \in \mathcal{F}_{\mathcal{H}}$ there exists W such that

$$\succ \;\equiv \succ_{(W,\mathcal{H})} \equiv TC\left(\bigcup_{A\in Var(\succ)} p_A\right)$$

$$p_A \equiv \{(o_1,o_2) \mid o_1.A \succ_A o_2.A\} \cap \approx_{\mathcal{A}-W_A-\{A\}}$$

We prove it by induction in the sizes of $\mathcal{H}$ and the corresponding $\mathcal{A}$.

BASE STEP. Let $\mathcal{H} = \{>_A\}$ and $\mathcal{A} = \{A\}$. Then $\mathcal{F}_{\mathcal{H}}$ consists of a single p-skyline relation which is induced by $>_A$. Take $W_A = \emptyset$. Then

$$\succ = \succ_{(W,\mathcal{H})} \equiv TC(p_A)$$

$$p_A \equiv \{(o_1,o_2) \mid o_1.A >_A o_2.A\} \cap \approx_{\mathcal{A}-W_A-\{A\}}$$

INDUCTIVE STEP. Now assume that the theorem holds for $\mathcal{H}$ and $\mathcal{A}$ of size up to n. Prove that it holds for $\mathcal{H}$ and $\mathcal{A}$ of size $n+1$. Let $\succ = \succ_1 \otimes \succ_2$ (the case of $\succ = \succ_1 \;\&\; \succ_2$ is similar). By the definition of p-skyline relations,

$$\succ \equiv (\succ_1 \cap \approx_{Var(\succ_2)}) \cup (\succ_2 \cap \approx_{Var(\succ_1)}) \cup (\succ_1 \cap \succ_2)$$

Thus, for two p-skyline relations $\succ_1$ and $\succ_2$ the inductive assumption implies that $\succ_1$ and $\succ_2$ can be represented by the structures $(W^1, \mathcal{H}^1)$ and $(W^2, \mathcal{H}^2)$ where $\mathcal{H}^1$ and $\mathcal{H}^2$ contain only the members of $\mathcal{H}$ for $Var(\succ_1)$ and $Var(\succ_2)$ correspondingly. That is,

$$\succ_1 = \succ_{(W^1,\mathcal{H})} \equiv TC(\bigcup_{A\in Var(\succ_1)} p_A^1) \tag{7.1}$$

$$\succ_2 = \succ_{(W^2,\mathcal{H})} \equiv TC(\bigcup_{A\in Var(\succ_2)} p_A^2) \tag{7.2}$$

where

$$p_A^1 \equiv \{(o_1,o_2) \mid o_1.A >_A o_2.A\} \cap \approx_{Var(\succ_1)-W_A^1-\{A\}} \tag{7.3}$$

$$p_A^2 \equiv \{(o_1,o_2) \mid o_1.A >_A o_2.A\} \cap \approx_{Var(\succ_2)-W_A^2-\{A\}} \tag{7.4}$$

Since $\succ$ is a p-skyline relation,

$$Var(\succ_1) \cap Var(\succ_2) = \emptyset. \tag{7.5}$$

(7.5), (7.1), and (7.2) imply

$$\succ \equiv TC\left(\bigcup_{A\in Var(\succ_1)} p_A^1\right) \cap \approx_{Var(\succ_2)} \cup\, TC\left(\bigcup_{A\in Var(\succ_2)} p_A^2\right) \cap \approx_{Var(\succ_1)} \cup$$
$$TC\left(\bigcup_{A\in Var(\succ_1)} p_A^1\right) \cap TC\left(\bigcup_{A\in Var(\succ_2)} p_A^2\right) \tag{7.6}$$

or equivalently

$$\succ \equiv TC\left(\bigcup_{A\in Var(\succ_1)} p_A^1 \cap \approx_{Var(\succ_2)}\right) \cup TC\left(\bigcup_{A\in Var(\succ_2)} p_A^2 \cap \approx_{Var(\succ_1)}\right) \cup$$
$$TC\left(\bigcup_{A\in Var(\succ_1)} p_A^1\right) \cap TC\left(\bigcup_{A\in Var(\succ_2)} p_A^2\right) \tag{7.7}$$

Construct the function W as follows

$$W_A = \begin{cases} W_A^1, & \text{if } A \in Var(\succ_1) \\ W_A^2, & \text{if } A \in Var(\succ_2) \end{cases}.$$

Take $\succ_{(W,\mathcal{H})}$ induced by $(W,\mathcal{H})$

$$\succ_{(W,\mathcal{H})} \equiv TC(\bigcup_{A\in\mathcal{A}} p_A^*) \tag{7.8}$$

for

$$p_A^* \equiv \{(o_1,o_2) \mid o_1.A >_A o_2.A\} \cap \approx_{\mathcal{A}-W_A-\{A\}} \tag{7.9}$$

We prove that $\succ_{(W,\mathcal{H})}$ is equivalent to $\succ$. Before going to the proof, notice that (7.7) can be rewritten as

$$\succ \equiv TC\left(\bigcup_{A\in Var(\succ_1)} p_A^*\right) \cup TC\left(\bigcup_{A\in Var(\succ_2)} p_A^*\right) \cup$$
$$TC\left(\bigcup_{A\in Var(\succ_1)} p_A^1\right) \cap TC\left(\bigcup_{A\in Var(\succ_2)} p_A^2\right) \tag{7.10}$$

1. Let $o \succ_{(W,\mathcal{H})} o'$. Let $(\Sigma_{o,o'}, \Psi_{o,o'})$ be any derivation sequence for $o \succ_{(W,\mathcal{H})} o'$. W.l.o.g.

let $\Psi_{o,o'} = (A_1, \dots, A_m)$, $\Sigma_{o,o'} = (o = o_1, o_2, \dots, o_m, o_{m+1} = o')$, and

$$p^*_{A_1}(o_1, o_2), p^*_{A_2}(o_2, o_3), \dots, p^*_{A_m}(o_m, o_{m+1}) \tag{7.11}$$

By construction, each attribute $A_i \in \Psi_{o,o'}$ is either in $Var(\succ_1)$ or $Var(\succ_2)$. For any such A_i, $p^*_{A_i}(o_i, o_{i+1})$ implies $o_i \succ o_{i+1}$ by (7.10). Therefore, (7.11) implies

$$o_1 \succ o_2, o_2 \succ o_3, ..., o_m \succ o_{m+1} \tag{7.12}$$

Transitivity of p-skyline relations implies $o_1 \succ o_{m+1}$, i.e. $o \succ o'$.

2. Let $o \succ o'$. Then (7.10) leads to three cases
   (a) $(o, o') \in TC\left(\bigcup_{A \in Var(\succ_1)} p^*_A\right)$. Then $o \succ_{(W,\mathcal{H})} o'$ by (7.8).
   (b) $(o, o') \in TC\left(\bigcup_{A \in Var(\succ_2)} p^*_A\right)$. Then $o \succ_{(W,\mathcal{H})} o'$ by the same reasoning.
   (c) $(o, o') \in TC\left(\bigcup_{A \in Var(\succ_1)} p^1_A\right) \cap TC\left(\bigcup_{A \in Var(\succ_2)} p^2_A\right)$.
   In this case, (7.5) implies that there is an object o'' whose values of $Var(\succ_2)$ are equal to those of o, and the values of $Var(\succ_1)$ are equal to those of o'. Then we have

$$(o, o'') \in TC\left(\bigcup_{A \in Var(\succ_1)} p^1_A\right) \cap \approx_{Var(\succ_2)}$$
$$(o'', o') \in TC\left(\bigcup_{A \in Var(\succ_2)} p^1_A\right) \cap \approx_{Var(\succ_1)}$$

or equivalently

$$(o,o'') \in TC\left(\bigcup_{A\in Var(\succ_1)} p^1_A \cap \approx_{Var(\succ_2)}\right)$$
$$(o'',o') \in TC\left(\bigcup_{A\in Var(\succ_2)} p^1_A \cap \approx_{Var(\succ_1)}\right)$$

which implies by (7.9) and (7.8)

$$o \succ_{(W,\mathcal{H})} o_1, o_1 \succ_{(W,\mathcal{H})} o'$$

The transitivity of $\succ_{(W,\mathcal{H})}$ implies $o \succ_{(W,\mathcal{H})} o'$. □

LEMMA 3.3. *Let for two p-skyline relations $\succ_1, \succ_2 \in \mathcal{F}_{\mathcal{H}}$, $(W^1,\mathcal{H})$ and $(W^2,\mathcal{H})$ be two structures inducing relations equal to $\succ_1$ and $\succ_2$ correspondingly. Let for some $A \in \mathcal{A}$, $W^1_A - W^2_A \neq \emptyset$. Then there is a pair $o,o' \in \mathcal{U}$ such that*

$$o \succ_1 o', o \not\succ_2 o'$$

PROOF

We construct two tuples o and o' such that $o \succ_{(W^1,\mathcal{H})} o'$ (and thus $o \succ_1 o'$), and $o \not\succ_{(W^2,\mathcal{H})} o'$ (and thus $o \not\succ_{(W^2,\mathcal{H})} o'$).

For every attribute $A_i \in \mathcal{A}$, pick two values $v_{A_i}, v'_{A_i} \in \mathcal{D}_{A_i}$ such that $v_{A_i} >_{A_i} v'_{A_i}$. Construct the tuples o and o' as follows:

$$o.A_i = \begin{cases} v_{A_i}, & \text{if } A_i = A, \\ v_{A_i}, & \text{if } A_i \in \mathcal{A} - \{A\} - W^1_A, \\ v'_{A_i}, & \text{otherwise}(A_i \in W^1_A) \end{cases}$$

$$o'.A_i = \begin{cases} v'_{A_i}, & \text{if } A_i = A, \\ v_{A_i}, & \text{if } A_i \in \mathcal{A} - \{A\} - W^1_A, \\ v_{A_i}, & \text{otherwise}(A_i \in W^1_A) \end{cases}$$

By construction, it is clear that

$$(o,o') \in \{(o_1,o_2) \mid o_1 \succ_A o_2\} \cap \approx_{\mathcal{A}-\{A\}-W^1_A}$$

and thus $o \succ_{(W^1,\mathcal{H})} o'$ and $o \succ_1 o'$. Now assume $o \succ_{(W^2,\mathcal{H})} o'$ (and thus $o \succ_2 o'$), i.e.

$$(o,o') = TC\left(\bigcup_{A_i \in \mathcal{A}} p_A\right) \tag{7.13}$$

where

$$p_{A_i} \equiv \{(o_1,o_2) \mid o_1 \succ_{A_i} o_2\} \cap \approx_{\mathcal{A}-W^2_{A_i}-\{A_i\}} . \tag{7.14}$$

(7.13) implies that there should exist a sequence of attributes $M = (A_{i_1},...,A_{i_p})$ and a sequence of tuples $o = o_1, o_2, ..., o_{p+1} = o'$ such that

$$p_{A_{i_1}}(o_1,o_2) \wedge ... \wedge p_{A_{i_p}}(o_p,o_{p+1}) \tag{7.15}$$

Note that by (7.14), o_{i_k} may be worse than $o_{i_{k+1}}$ in the values of $W_{A^2_{i_k}}$ only.

Prove that $M \subseteq W^2_A \cup \{A\}$. For the sake of contradiction, assume $M^- = M - (W^2_A \cup \{A\})$ is nonempty. Pick an element $A_{top} \in M^-$ which has no ancestors from M^- in $\Gamma_{\succ_2}$ (such an element exists due to acyclicity of $\Gamma_{\succ_2}$). Since $p_{A_{top}}$ is in the chain (7.15), we get

$$o.A_{top} >_{A_{top}} o'.A_{top} \tag{7.16}$$

By construction of o, o' that implies $A_{top} = A$ which is a contradiction. Thus, $M \subseteq W^2_A \cup \{A\}$.

Now pick any $B \in W^1_A - W^2_A$. By construction of o, o', $o'.B \succ_B o.B$. Therefore, there is $C \in M$ such that $B \in W^2_C$. The transitivity of $\Gamma_{\succ_2}$ implies $B \in W^2_A$ which contradicts the initial assumption about B. Thus $o \not\succ_{(W^2,\mathcal{H})} o'$ and $o \not\succ_2 o'$ □

THEOREM 3.6. *Let $\succ$ be a p-skyline relation with the p-graph $\Gamma_\succ$. Let also $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$ be disjoint node sets of $\Gamma_\succ$. Let the subgraphs of $\Gamma_\succ$ induced by those node sets be singletons or unions of at least two disjoint subgraphs. Then*

$$(\mathbf{A},\mathbf{B}) \in \Gamma_\succ \wedge (\mathbf{C},\mathbf{D}) \in \Gamma_\succ \wedge (\mathbf{C},\mathbf{B}) \in \Gamma_\succ \Rightarrow (\mathbf{C},\mathbf{A}) \in \Gamma_\succ \vee (\mathbf{A},\mathbf{D}) \in \Gamma_\succ \vee (\mathbf{D},\mathbf{B}) \in \Gamma_\succ$$

PROOF

Prove the theorem by contradiction. Let

$$(\mathbf{A},\mathbf{B}) \in \Gamma_\succ \wedge (\mathbf{C},\mathbf{D}) \in \Gamma_\succ \wedge (\mathbf{C},\mathbf{B}) \in \Gamma_\succ \wedge (\mathbf{C},\mathbf{A}) \notin \Gamma_\succ \wedge (\mathbf{A},\mathbf{D}) \notin \Gamma_\succ \wedge (\mathbf{D},\mathbf{B}) \notin \Gamma_\succ$$

The second part is equivalent to the following.

$$\exists C \in \mathbf{C}, A_1, A_2 \in \mathbf{A}, D_1, D_2 \in \mathbf{D}, B \in \mathbf{B}($$

$$(C, A_2) \notin \Gamma_\succ \wedge \qquad \text{(C-A2)}$$

$$(A_1, D_1) \notin \Gamma_\succ \wedge \qquad \text{(A1-D1)}$$

$$(D_2, B) \notin \Gamma_\succ) \qquad \text{(D2-B)}$$

and

$$(A_1, B) \in \Gamma_{\succ} \tag{A1-B}$$

$$(A_2, B) \in \Gamma_{\succ} \tag{A2-B}$$

$$(C, D_1) \in \Gamma_{\succ} \tag{C-D1}$$

$$(C, D_2) \in \Gamma_{\succ} \tag{C-D2}$$

Note that the fact that the subgraphs of $\Gamma_{\succ}$ induced by **A**, **B**, **C**, **D** are singletons or unions of at least two disjoint subgraphs, implies the following four cases for A_1 and A_2

$$\Gamma_{\succ} \models A_1 \sim A_2 \tag{Case A1}$$

$$(A_1, A_2) \in \Gamma_{\succ} \wedge \exists A_3 \in \mathbf{A} \,.\, \Gamma_{\succ} \models A_1 \sim A_3 \wedge \Gamma_{\succ} \models A_2 \sim A_3 \tag{Case A2}$$

$$A_2 \rightarrow A_1 \Gamma_{\succ} \wedge \exists A_3 \in \mathbf{A} \,.\, \Gamma_{\succ} \models A_1 \sim A_3 \wedge \Gamma_{\succ} \models A_2 \sim A_3 \tag{Case A3}$$

$$A_1 \equiv A_2 \tag{Case A4}$$

Similarly, we have four cases for D_1, D_2.

$$\Gamma_{\succ} \models D_1 \sim D_2 \tag{Case D1}$$

$$(D_1, D_2) \in \Gamma_{\succ} \wedge \exists D_3 \in \mathbf{D} \,.\, \Gamma_{\succ} \models D_1 \sim D_3 \wedge \Gamma_{\succ} \models D_2 \sim D_3 \tag{Case D2}$$

$$(D_2, D_1) \in \Gamma_{\succ} \wedge \exists D_3 \in \mathbf{D} \,.\, \Gamma_{\succ} \models D_1 \sim D_3 \wedge \Gamma_{\succ} \models D_2 \sim D_3 \tag{Case D3}$$

$$D_1 \equiv D_2 \tag{Case D4}$$

Therefore, totally we have sixteen different cases, and we need to show that all of them lead to contradictions. One can show that all of them contradict the `Envelope` property. We demonstrate it for the case (A3-D2), while the other cases are handled similarly. In Figure 7-1, we show instances of the `Envelope` property. Recall that the `Envelope` property says that if a graph has three certain edges, it must have at least one of other three edges. The instances we show below lead to only one possible edge while the other two violate some conditions above. The violated condition is shown below each corresponding edge. Finally, we show that there is an unsatisfiable instance of the `Envelope` property.

We have automatically tested the other fifteen cases and showed that similar contradictions can be derived for them, too. □

`Envelope` **condition**	**first edge**	**second edge**	**third edge**
(A_2,B), (C,D_2), (C,B)	(D_2,B) (D2-B)	(A_2,D_2)	(C,A_2) (C-A2)
(A_2,D_2), (C,D_3), (C,D_2)	(D_3,D_2) (D3 $\sim$ D2)	(C,A_2) (C-A2)	(A_2,D_3)
(A_3,B), (A_2,D_2), (A_2,B)	(D_2,B) (D2-B)	(A_2,A_3) (A2 $\sim$ A3)	(A_3,D_2)
(A_3,D_2), (A_2,D_3), (A_2,D_2)	(A_3,D_3)	(D_3,D_2) (D3 $\sim$ D2)	(A_2,A_3) (A2-A3)
(A_2,D_3), (C,D_1), (C,D_3)	(A_2,D_1)	(C,A_2) (C-A2)	(D_1,D_3) (D1 $\sim$ D3)
(D_1,D_2), (A_3,D_3), (A_3,D_2)	(D_3,D_2) (D3 $\sim$ D2)	(A_3,D_1)	(D_1,D_3) (D1 $\sim$ D3)
(A_3,D_1), (A_2,A_1), (A_2,D_1)	(A_2,A_3) (A2 $\sim$ A3)	(A_1,D_1) (A1-D1)	(A_3,A_1) (A3 $\sim$ A1)

Figure 7-1: *Case $A3$-$D2$*

LEMMA 3.4. *Let $\succ_{ext}$ be a full p-skyline extension of $\succ \in \mathcal{F}_{\mathcal{H}}$, and $T_{\succ}$ be a normalized syntax tree of $\succ$. Let also (C_1,C_2) be a frontier pair of $T_{\succ}$ w.r.t. $T_{\succ_{ext}}$. Denote the top and* *the bottom components of C_1 as A_1, B_1, and the top and the bottom components of C_2 as A_2, B_2. Then*

$$(Var(A_1), Var(B_2)) \in \Gamma_{\succ_{ext}} \vee (Var(A_2), Var(B_1)) \in \Gamma_{\succ_{ext}}$$

PROOF

Since (C_1, C_2) is a frontier pair of $T_{\succ}$ w.r.t. $T_{\succ_{ext}}$, there are $X \in Var(C_1)$ and $Y \in Var(C_2)$ such that

$$(X, Y) \in \Gamma_{\succ_{ext}}$$

Note that we have the following cases for $X \in Var(C_1)$

ϕ_1	$Var(C_1) = \{X\}$, i.e. $(C_1 = A_1 = B_1)$
ϕ_2	$C_1 = (A_1 \ \& \ \ldots \ \& \ B_1), X \notin Var(A_1)$
ϕ_3	$C_1 = (A_1 \ \& \ \ldots \ \& \ B_1), Var(A_1) = \{X\}$
ϕ_4	$C_1 = (A_1 \ \& \ \ldots \ \& \ B_1)$, $A_1 = A_1^1 \otimes A_1^2 \ldots, X \in Var(A_1^1)$

and for $Y \in Var(C_2)$

λ_1	$Var(C_2) = \{Y\}$, i.e. $(C_2 = A_2 = B_2)$
λ_2	$C_2 = (A_2 \ \& \ \ldots \ \& \ B_2), Y \notin Var(B_2)$
λ_3	$C_2 = (A_2 \ \& \ \ldots \ \& \ B_2), Var(B_2) = \{Y\}$
λ_4	$C_2 = (A_2 \ \& \ \ldots \ \& \ B_2)$ $B_2 = B_2^1 \otimes B_2^2 \ldots, Y \in Var(B_2^1)$.

The cases ϕ_1, ϕ_2, ϕ_3 imply that $Var(A_1) = \{X\}$ or $(Var(A_1), X) \in \Gamma_{\succ_{ext}}$ and as a result $(Var(A_1), Y) \in \Gamma_{\succ_{ext}}$ by transitivity of $\Gamma_{\succ_{ext}}$. Similarly, the cases $\lambda_1, \lambda_2, \lambda_3$ imply $Var(B_2) =$

$\{Y\}$ or $(Var(B_2),Y) \in \Gamma_{\succ_{ext}}$. Thus any combination of these cases imply $(Var(A_1),Var(B_2)) \in \Gamma_{\succ_{ext}}$. Now consider the other combinations of the cases. All of them are handled similar to the case (ϕ_4, λ_4), so we consider it in detail.

Take the case λ_4. Take any $Y' \in Var(B_2) - Var(B_2^1)$ and apply `GeneralEnvelope` to $\Gamma_{\succ_{ext}}$:

$$(Var(A_2),Y) \in \Gamma_{\succ_{ext}} \wedge (Var(A_2),Y) \in \Gamma_{\succ_{ext}} \wedge (X,Y) \in \Gamma_{\succ_{ext}}$$

which implies

$$(Var(A_2),X) \in \Gamma_{\succ_{ext}} \vee (X,Y') \in \Gamma_{\succ_{ext}} \vee (Y',Y) \in \Gamma_{\succ_{ext}}.$$

Clearly, $(Y,Y') \notin \Gamma_{\succ_{ext}}$ follows from Proposition 3.4 and the fact that the subgraphs of $\Gamma_{\succ_{ext}}$ and $\Gamma_{\succ}$ induced by $Var(C_2)$ are the same. Moreover, $(Var(A_2),X) \in \Gamma_{\succ_{ext}}$ along with the fact $(X,Var(B_1)) \in \Gamma_{\succ_{ext}}$ implies $(Var(A_2),Var(B_1)) \in \Gamma_{\succ_{ext}}$ which is what we need. As a result, we get that either $(Var(A_2),Var(B_1)) \in \Gamma_{\succ_{ext}}$ or $(X,Y') \in \Gamma_{\succ_{ext}}$ for all $Y' \in Var(B_2) - Var(B_2^1)$. Now, take every $Y'' \in Var(B_2^1)$. For any such Y'' we have $(Y',Y'') \notin \Gamma_{\succ_{ext}}$ by Proposition 3.4. As a result, we get the same `GeneralEnvelope` property as above with Y replaced with Y' and Y' with Y''. It implies that either $(Var(A_2),Var(B_1)) \in \Gamma_{\succ_{ext}}$ or $(X,Y'') \in \Gamma_{\succ_{ext}}$ for all $Y'' \in Var(B_2^1)$. Therefore, $(X,Var(B_2)) \in \Gamma_{\succ_{ext}}$ or $(Var(A_2),Var(B_1)) \in \Gamma_{\succ_{ext}}$.

Elaborating the case ϕ_4 as above gives that

$$(Var(A_2),Var(B_1)) \in \Gamma_{\succ_{ext}} \ \vee \ (Var(A_1),Y) \in \Gamma_{\succ_{ext}}.$$

Combining these two results gives

$$(Var(A_1), Var(B_2)) \in \Gamma_{\succ_{ext}} \vee (Var(A_2), Var(B_1)) \in \Gamma_{\succ_{ext}}.$$

□

LEMMA 4.1. *Take a full p-skyline relation $\succ$, a disjoint subsets G and W of a set of objects O. Then the next two operations can be done in polynomial time.*

1. *Verifying if $\succ$ is optimal favoring G and disfavoring W in O;*
2. *Computing an optimal p-skyline relation $\succ' \in \mathcal{F}_{\mathcal{H}}$ favoring G and disfavoring W in O which is an extension of $\succ$*

PROOF

To check if $\succ$ favors G and disfavors W in O, we need to compute $w_{\succ}(O)$, check $G \subseteq w_{\succ}(O)$, and verify that for every $o \in W$, there is $o' \in G$ such that $o' \succ o$. It can clearly be done in polynomial time. If any of these conditions fails, then $\succ$ is obviously not optimal. Otherwise, we need to check if every its minimal extension favors G and disfavors W. Note that since $\succ$ disfavors W in O, any its extension also disfavors W in O. Hence, $\succ$ is not optimal if at least one minimal extension favors G in O, and it is optimal otherwise. Corollaries 3.5 and 3.6 imply that all minimal extension of $\succ$ can be computed in polynomial time.

To construct an optimal extension $\succ'$ of $\succ$, we can take $\succ$, construct its every minimal extension and verify if at least one of them favors G in O. If any of them does, we pick it and repeat the same procedure for it. We do it until for some $\succ'$ its every minimal extension does not favor G in O. This implies that $\succ'$ is an optimal p-skyline relation favoring G and disfavoring W in O. Moreover, $\succ'$ is a superset of $\succ$ by construction. Corollaries 3.5, 3.6, and 3.7 imply that such a computation can be done in polynomial time. □

THEOREM 5.2. *Let N be an HCP-net. Then for two different tuples $o, o' \in \mathcal{U}$. Let $\mathbf{Diff}(o,o') \subseteq \mathcal{A}$ be the set of attributes in which o and o' are different, and $\mathbf{Top}(o,o')$ be the set of the top most attributes of $\mathbf{Diff}(o,o')$ in Υ_N, i.e. those which have no ancestors in $\mathbf{Diff}(o,o')$. Then the following are equivalent:*

1. $o \succ_N o'$;
2. $\forall A \in \mathbf{Top}(o,o') \; . \; CPT^r_A(o,o')$
3. $\forall A \in \mathbf{Diff}(o,o') \; \exists B \in Anc\text{-}self_{\Upsilon_N}(A) \; . \; CPT^r_B(o,o')$

PROOF

$\boxed{1 \Leftrightarrow 2}$ First, we prove that 2 implies 1. We construct (Σ, Ψ) where $\Psi = \mathbf{Top}(o,o') = \{A_{i_1}, \ldots, A_{i_k}\}$ and $\Sigma = \{o_1, \ldots, o_{k+1})$ which is a derivation sequence for $o \succ_N o'$.

Let o_1 be equal to o. For every $j \in [2,k]$, we construct o_j as follows: 1) set the values of the attributes $Desc\text{-}self_{\Upsilon_N}(A_{i_{j-1}})$ of o_j to the corresponding values of o', and 2) set the values of the other attributes to the corresponding values of o_{j-1}.

This construction has the following properties. For every o_{j-1} and o_j: 1) the values of $Anc_{\Upsilon_N}(A_{i_j})$ are equal in them and in o and o', 2) the values of $Sibl_{\Upsilon_N}(A_{i_j})$ are equal in o_{j-1} and o_j, and 3) $o_{j-1}.A_{i_j} = o.A_{i_j}$ and $o_j.A_{i_j} = o'.A_{i_j}$. Therefore, $CPT^r_{A_j}(o,o')$ implies $CPT^*_{A_j}(o_{j-1}, o_j)$. Moreover, because $\mathbf{Top}(o,o')$ are the topmost nodes in $\mathbf{Diff}(o,o')$, $o_{k+1} = o'$. Therefore, (Σ, Ψ) is a derivation sequence of $o \succ_N o'$.

Now we prove that 1 implies 2. Let $o \succ_N o'$ and $(\Sigma_{o,o'}, \Psi_{o,o'})$ be a derivation sequence for $o \succ_N o'$: $\Sigma_{o,o'} = \{o = o_1, \ldots, o_{k+1} = o'\}$ and $\Psi_{o,o'} = \{A_{i_1}, \ldots, A_{i_k}\}$. Due to the HCP-net semantics, we have that for every $j \in [1,k]$

$$\mathbf{Diff}(o_j, o_{j+1}) \subseteq Desc\text{-}self_{\Upsilon_N}(A_{i_j}).$$

Hence,

$$\mathbf{Top}(o,o') \subseteq \mathbf{Diff}(o,o') \subseteq \bigcup_{B \in \Psi_{o,o'}} \textit{Desc-self}_{\Upsilon_N}(B)$$

We show that

$$\text{no member } B \text{ of } \mathbf{Top}(o,o') \text{ has an ancestor in } \Psi_{o,o'}. \tag{7.17}$$

Note that if some $B \in \mathbf{Top}(o,o')$ has an ancestor $C \in \Psi_{o,o'}$ then CPT^r_C is not an SPO or there is an ancestor $D \in \Psi_{o,o'}$ of C (which is also an ancestor of B) and so on. That contradicts to the finiteness and SPO of Υ_N. Hence, 7.17 holds, and all tuples in $\Sigma_{o,o'}$ are equal in the ancestors of all the members of $\mathbf{Top}(o,o')$.

Now take any $A \in \mathbf{Top}(o,o')$. Take the sequence of the values of A in the tuples $\Sigma_{o,o'}$. Then for every pair a, a' in this sequence (s.t. a precedes a'), a is preferred to a' according to R_q of the same $q \in \Phi_A$ for all pairs (due to the argument above). By the transitivity of R_q, $R_q(o.A, o'.A)$. Therefore, $CPT^r_A(o,o')$.

$\boxed{2 \Leftrightarrow 3}$ First, let $\forall A \in \mathbf{Top}(o,o') \,.\, CPT^r_A(o,o')$. For contradiction, assume

$$\exists B \in \mathbf{Diff}(o,o') \; \forall C \in \textit{Anc-self}_{\Upsilon_N}(B) \,.\, \neg CPT^r_C(o,o'). \tag{7.18}$$

Then any topmost element C of $\textit{Anc-self}_{\Upsilon_N}(B) \cap \mathbf{Diff}(o,o')$ is also in $\mathbf{Top}(o,o')$. Then (7.18) implies $\neg CPT^r_C(o,o')$ which contradicts the initial assumption.

Now let

$$\forall B \in \mathbf{Diff}(o,o') \; \exists C \in \textit{Anc-self}_{\Upsilon_N}(B) \,.\, CPT^r_C(o,o'). \tag{7.19}$$

For the sake of contradiction, assume

$$\exists A \in \mathbf{Top}(o,o') \,.\, \neg CPT^r_A(o,o'). \tag{7.20}$$

Then (7.19) and (7.20) imply that there is $C \in Anc_{\Upsilon_N}(A)$ such that $CPT^r_C(o,o')$. That implies $o.C \neq o'.C$, i.e. $C \in \mathbf{Diff}(o,o')$. Then the fact that C is an ancestor of A in Υ_A implies $A \notin \mathbf{Top}(o,o')$ which is a contradiction. $\square$